INVERSE PROBLEM FOR A TIME-DEPENDENT CONVECTION-DIFFUSION EQUATION IN ADMISSIBLE GEOMETRIES

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Abstract. We consider a partial data inverse problem for a time-dependent convection-diffusion equation on an admissible manifold. We prove that the time-dependent convection term and time-dependent density can be recovered uniquely modulo a known gauge invariance. There have been several works on inverse problems related to the steady state convection-diffusion operator in Euclidean as well as in Riemannian geometry settings; however, inverse problems related to time-dependent convection-diffusion equation on a manifold are not studied in the prior works, which is the main aim of this paper. In fact, to the best of our knowledge, the problem studied here is the first work related to a partial data inverse problem for recovering both first and zeroth-order time-dependent perturbations of evolution equations in the Riemannian geometry setting.

Keywords: Inverse problems, time-dependent coefficients, convection-diffusion equation, partial boundary data, admissible manifold, Carleman estimates, geometric optics solutions.

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1. Introduction and statement of main result

The paper deals with a partial data inverse problem related to a convection-diffusion equation on $M_T := (0,T)\times M$ where $0<T<\infty$ and (M,g) is a smooth $n-$dimensional $(n\geq 2)$ Riemannian manifold having smooth boundary ∂M. We denote by $\Sigma := (0,T)\times\partial M$ as the lateral boundary of M_T and $\partial M_T := \Sigma\cup(\{0\}\times M)\cup(\{T\}\times M)$ the topological boundary of M_T. We also denote by TM and T^*M the tangent and cotangent bundle of M. For a convection term $A\in W^{1,\infty}(M_T;T^*M)$ given by $A(t,x) := \sum_{j=1}^{n} A_j(t,x)dx^j$ in local coordinates $x_1,x_2,\dots,x_n$ of manifold M and density $q\in L^\infty(M_T)$, the initial boundary value problem (IBVP) for the convection-diffusion equation on M_T is modeled by the following IBVP for second order linear parabolic partial differential equation (PDE)

$$\begin{cases} \Big[\partial_t - \sum_{j,k=1}^{n} \frac{1}{\sqrt{|g|}}\Big(\partial_{x_j}+A_j\Big)\Big(\sqrt{|g|}g^{jk}\left(\partial_{x_k}+A_k\right)\Big)+q\Big]u(t,x)=0,\ (t,x)\in M_T\\ u(0,x)=\phi(x),\ x\in M\\ u(t,x)=f(t,x),\ (t,x)\in\Sigma \end{cases}\tag{1.1}$$

where $g^{-1} := \left((g^{ij})\right)_{1\leq i,j\leq n}$ denote the inverse of metric tensor $g := ((g_{ij}))_{1\leq i,j\leq n}$, $|g| = \det(g)$ and the initial value ϕ and the Dirichlet data f are assumed to be non-zero. Throughout this article, we denote by $\mathcal{L}_{A,q}$ the following operator

$$\mathcal{L}_{A,q} := \partial_t - \sum_{j,k=1}^{n}\frac{1}{\sqrt{|g|}}\Big(\partial_{x_j}+A_j(t,x)\Big)\Big(\sqrt{|g|}g^{jk}\left(\partial_{x_k}+A_k(t,x)\right)\Big)+q(t,x).\tag{1.2}$$

In this paper, we are interested in determining the convection term A and density coefficient q from the boundary measurements of the solution. To define the boundary operators, we need to have the existence and uniqueness of a solution to the forward problem for IBVP given by (1.1).

Motivated by [15, 49], we define the following spaces

$$\mathcal{K}_0 := \{(f|_{t=0}, f|_{\Sigma}) : \; f \in H^1(0,T,H^{-1}(M)) \cap L^2(0,T;H^1(M))\} \;\text{ and}$$
$$\mathcal{K}_T := \{(f|_{t=T}, f|_{\Sigma}) : \; f \in H^1(M_T)\}$$

where we refer to [25] for the definition of function spaces $H^m(0,T;H^k(M))$, for $k,m \in \mathbb{R}$. Now for $(\phi, f) \in \mathcal{K}_0$, it can be shown by following arguments from [15, 25, 43] that there exists a unique solution $u \in H^1(0,T,H^{-1}(M)) \cap L^2(0,T;H^1(M))$ of IBVP (1.1). Based on the existence and uniqueness of solution and following [15, 40, 49], we observe that for any solution $u \in H^1(0,T,H^{-1}(M)) \cap L^2(0,T;H^1(M))$ the operator $\mathcal{N}_{A,q}u$ given by

$$\begin{aligned}\langle \mathcal{N}_{A,q}u, w|_{\partial M_T^*}\rangle := &\int_{M_T} \left(-u\partial_t\overline{w} + \langle \nabla_g u, \nabla_g \overline{w}\rangle_g + 2u\langle A, \nabla_g\overline{w}\rangle_g + (\delta_g A)u\overline{w} - |A|_g^2 u\overline{w} + qu\overline{w}\right)\, dV_g dt\\ &- \int_M u(0,x)\overline{w}(0,x)dV_g\end{aligned}$$

is well-defined for all $w \in H^1(M_T)$ where $\partial M_T^* := (\{T\}\times M)\cup\Sigma$. Now if we assume the sufficient regularity on the coefficients A, q and Dirichlet data f, then as shown in [15] the operator $\mathcal{N}_{A,q}u$ is given by

$$\mathcal{N}_{A,q}u := \left(u|_{t=T}, \left[\partial_\nu u(t,x) + 2\langle \nu, A\rangle_g(t,x)u(t,x)\right]\Big|_{\Sigma}\right)$$

where ν stands for the outward unit normal vector to ∂M and u solves the IBVP given by (1.1). This motivates us to define our input-output operator $\Lambda_{A,q} : \mathcal{K}_0 \to \mathcal{K}_T^*$ by

$$\Lambda_{A,q}(\phi, f) := \mathcal{N}_{A,q}u \tag{1.3}$$

where $\mathcal{K}_T^*$ stands for dual of $\mathcal{K}_T$ and u is solution to the IBVP (1.1) when the initial data is ϕ and Dirichlet boundary data equal to f.

This work is concerned with the determination of time-dependent coefficients A and q appearing in (1.1) using the measurements of the input-output operator $\Lambda_{A,q}$ on a proper subset of Σ for the case when (M,g) is an admissible manifold whereby an admissible manifold, we mean the following.

Definition 1.1. (Admissible manifold [11, 29]) We say that a compact Riemannian manifold (M,g) of dimension $n \geq 2$ with boundary ∂M, is admissible if M is orientable and (M,g) is a submanifold of $\mathbb{R} \times (\text{int}(M_0), g_0)$ where (M_0, g_0) is a compact, simply connected Riemannian manifold with boundary ∂M_0 which is strictly convex in the sense of the second fundamental form and M_0 has no conjugate points.

Some examples of admissible manifolds are the following (for more examples, see [29]):

1. Bounded domains in Euclidean space.
2. Any bounded domain M in $\mathbb{R}^n$, endowed with a metric which in some coordinates has the form
$$g(x_1, x') = \begin{pmatrix} 1 & 0 \\ 0 & g_0(x') \end{pmatrix},$$
is admissible.

The assumptions for admissible manifolds are applied in various contexts, such as selecting a limiting Carleman weight, calculating Carleman estimates, proving the injectivity of the geodesic ray transform, and in many other places.

In order to state the main result of this article, we first need to specify the subset of ∂M where the measurements are given. Now if we write $x \in M$, as $x := (x_1, x') \in \mathbb{R} \times M_0$ and $\varphi(x) := x_1$ then ∂M can be decomposed into the two parts given by

$$\partial M_+ := \{x \in \partial M : \; \partial_\nu \varphi(x) > 0\} \text{ and } \partial M_- := \{x \in \partial M : \; \partial_\nu\varphi(x) \leq 0\}$$

where $\nu(x)$ stands for outward unit normal to ∂M at $x \in \partial M$ and $\partial_\nu\varphi$ denote the normal derivative of φ with respect to the metric g. In this paper, we will be assuming that our boundary measurements are given on slightly bigger than half of ∂M. To specify this portion of ∂M, we take $\epsilon > 0$ small enough and define $\partial M_{\pm,\epsilon/2}$ by

$$\partial M_{+,\epsilon/2} := \left\{x \in \partial M : \ \partial_\nu \varphi(x) \geq \frac{\epsilon}{2}\right\} \text{ and } \partial M_{-,\epsilon/2} := \left\{x \in \partial M : \ \partial_\nu \varphi(x) < \frac{\epsilon}{2}\right\} \tag{1.4}$$

as $\partial M_{-,\epsilon/2}$ is the small enough open neighborhood of ∂M_-. We denote the corresponding lateral part of Σ by $\Sigma_+ := (0,T)\times\partial M_+$, $\Sigma_{+,\epsilon/2} := (0,T)\times\partial M_{+,\epsilon/2}$, $\Sigma_- := (0,T)\times\partial M_-$ and $\Sigma_{-,\epsilon/2} := (0,T)\times\partial M_{-,\epsilon/2}$. We also denote $\partial M^*_{T_\pm} := (\{T\}\times M)\cup\Sigma_\pm$ and $\partial M^*_{T_{\pm,\epsilon/2}} := (\{T\}\times M)\cup\Sigma_{\pm,\epsilon/2}$. Now, using these notations, we define the partial input-output operator by

$$\Lambda^{partial}_{A,q}(\phi, f) := \mathcal{N}_{A,q} u|_{\partial M^*_{T_{-,\epsilon/2}}}. \tag{1.5}$$

Our aim in this article is to recover A and q uniquely from the knowledge of $\Lambda^{partial}_{A,q}$ however, due to gauge invariance, it is impossible to recover these coefficients fully. Since this is the first work related to the time-dependent convection-diffusion equation on manifolds therefore before stating the main result of this paper, we first provide quick proof of the gauge invariance associated with our problem. In the Euclidean setting, this has been well observed in prior works; see, for example, [15, 49] and references therein.

Definition 1.2. (Gauge Invariance) Let $A^{(i)} \in W^{1,\infty}(M_T)$ and $q_i \in L^\infty(M_T)$ for $i = 1, 2$. We say $(A^{(1)}, q_1)$ and $(A^{(2)}, q_2)$ are gauge equivalent if there exists $\Psi \in W^{2,\infty}_0(M_T)$ such that

$$A^{(2)}(t,x) = A^{(1)}(t,x) - \nabla_g\Psi(t,x) \text{ and } q_2(t,x) = q_1(t,x) - \partial_t\Psi(t,x), \text{ for } (t,x) \in M_T.$$

Proposition 1.3. *Suppose $u_1(t,x)$ is a solution to the following IBVP*

$$\begin{cases} \left[\partial_t - \sum_{j,k=1}^n \frac{1}{\sqrt{|g|}}\left(\partial_{x_j} + A^{(1)}_j\right)\left(\sqrt{|g|}g^{jk}\left(\partial_{x_k} + A^{(1)}_k\right)\right) + q_1\right]u_1(t,x) = 0, \ (t,x) \in M_T \\ u_1(0,x) = \phi(x), \ x \in M \\ u_1(t,x) = f(t,x), \ (t,x) \in \Sigma \end{cases} \tag{1.6}$$

and $\Psi \in W^{2,\infty}_0(M_T)$, then $u_2(t,x) = e^{\Psi(t,x)}u_1(t,x)$ satisfies the following IBVP

$$\begin{cases} \left[\partial_t - \sum_{j,k=1}^n \frac{1}{\sqrt{|g|}}\left(\partial_{x_j} + A^{(2)}_j\right)\left(\sqrt{|g|}g^{jk}\left(\partial_{x_k} + A^{(2)}_k\right)\right) + q_2\right]u_2(t,x) = 0, \ (t,x) \in M_T \\ u_2(0,x) = \phi(x), \ x \in M \\ u_2(t,x) = f(t,x), \ (t,x) \in \Sigma \end{cases} \tag{1.7}$$

where $A^{(2)}(t,x) = A^{(1)}(t,x) - \nabla_g\Psi(t,x)$ and $q_2(t,x) = q_1(t,x) - \partial_t\Psi(t,x)$. Now if $\Lambda_{A^{(i)},q_i}$ for $i = 1, 2$, are the input-output operators associated with u_i and defined by (1.3) *then*

$$\Lambda_{A^{(1)},q_1}(\phi, f) = \Lambda_{A^{(2)},q_2}(\phi, f), \text{ for all } (\phi, f) \in \mathcal{K}_0.$$

Proof. Substituting $u_1(t,x) = e^{-\Psi(t,x)}u_2(t,x)$ in Equation (1.6), from simple computations, we get

$$0 = \mathcal{L}_{A^{(1)},q_1}(e^{-\Psi(t,x)}u_2(t,x)) = \mathcal{L}_{A^{(1)}-\nabla_g\Psi, q_1-\partial_t\Psi}u_2(t,x) = \mathcal{L}_{A^{(2)},q_2}u_2(t,x), \qquad (t,x) \in M_T$$

and

$$u_2(0,x) = e^{\Psi(0,x)}u_1(0,x) = \phi(x), \quad x \in M,$$
$$u_2(t,x) = e^{\Psi(t,x)}u_1(t,x) = f(t,x), \quad (t,x) \in \Sigma.$$

Hence u_2 solves (1.7). Also, we have that

$$u_2(T,x) = e^{\Psi(T,x)}u_1(T,x) = u_1(T,x), \quad x \in M, \quad \partial_\nu u_2\Big|_\Sigma = \left(e^\Psi(\partial_\nu\Psi u_1 + \partial_\nu u_1)\right)\Big|_\Sigma = \partial_\nu u_1\Big|_\Sigma,$$

$$\text{and } \left(\langle\nu, A^{(2)}\rangle_g u_2\right)\Big|_\Sigma = \left(\langle\nu, A^{(1)} - \nabla_g\Psi\rangle_g e^\Psi u_1\right)\Big|_\Sigma = \left(\langle\nu, A^{(1)}\rangle_g u_1\right)\Big|_\Sigma$$

where in the above equations, we have used the fact that $\Psi \in W_0^{2,\infty}(M_T)$. Thus combining the above equations together with (1.3) we get

$$\Lambda_{A^{(1)},q_1}(\phi, f) = \Lambda_{A^{(2)},q_2}(\phi, f), \text{ for all } (\phi, f) \in \mathcal{K}_0.$$

□

With this preparation, we are ready to state the main result of this paper as follows.

Theorem 1.4. *Let (M, g) be an admissible manifold. Let $A^{(i)} \in W^{1,\infty}(M_T; T^*(M))$ for $i = 1, 2$ given by $A^{(i)}(t, x) = \sum_{j=1}^{n} A_j^{(i)}(t, x)dx^j$, in local coordinates on (M, g) and $q_i \in L^\infty(M_T)$ for $i = 1, 2$. Suppose u_i for $i = 1, 2$, is solution to (1.1) when $(A, q) = (A^{(i)}, q_i)$ for $i = 1, 2$ and $\Lambda^{partial}_{A^{(i)},q_i}$ are input-output operator given by (1.5) corresponding to u_i for $i = 1, 2$. Now for $\epsilon > 0$ small enough if*

$$\Lambda^{partial}_{A^{(1)},q_1}(\phi, f) = \Lambda^{partial}_{A^{(2)},q_2}(\phi, f), \text{ for all } (\phi, f) \in \mathcal{K}_0 \tag{1.8}$$

then there exists a function $\Psi \in W_0^{2,\infty}(M_T)$ such that

$$A^{(1)}(t, x) - A^{(2)}(t, x) = \nabla_g \Psi(t, x) \text{ and } q_1(t, x) - q_2(t, x) = \partial_t \Psi(t, x), \text{ for } (t, x) \in M_T$$

provided $A^{(1)}(t, x) = A^{(2)}(t, x)$, for $(t, x) \in \Sigma$.

Remark 1.5.
(1) Observe that the measurement data used in Theorem 1.4 is an input-output map, unlike the usual Dirichlet to Neumann (DN) map used in the Euclidean setting. This is due to the fact that in our boundary Carleman estimate stated in Theorem 2.1, we do not have the estimate on weighted L^2 norm of solution u at $t = T$ which is because of the choice of weight function $\varphi(t, x) := \lambda^2\beta^2 t + \lambda x_1$ where $\beta \in (0, 1)$ while in Euclidean setting one can actually take $\beta = 1$ which helped one to the Carleman estimate with a bound on the weighted L^2 norm of the solution u at $t = T$. We refer to [15, 51] for details about it. However, if we assume the coefficients are small enough then we can obtain the boundary Carleman estimate (see Theorem 3.1 in [49]) with a bound on the weighted L^2 norm of the solution u at $t = T$ and can determine the coefficients from the knowledge of DN map measured on a suitable subset of Σ.
(2) We observe that because of gauge invariance proved in Proposition 1.3, it is impossible to prove that $A^{(1)} = A^{(2)}$ and $q_1 = q_2$ in M_T from the given hypothesis of Theorem 1.4. As far as the uniqueness issue is concerned, gauge invariance guarantees that the result obtained in Theorem 1.4 is optimal.
(3) Unique recovery of A and q is also possible with some extra conditions on the unknown vector field A. For instance, if A is divergence-free, that is, $\delta_g A = 0$ in M_T, then one can recover both A and q uniquely in M_T. This divergence-free condition has been exploited in earlier works to get the full recovery, please refer [49, 51].
(4) On the other side, if we assume that the vector field A is time-independent, then recovery of A is possible up to a potential of the form $\nabla_g \Psi(x)$ and q can be fully recovered in M_T. The proof follows similarly as we have done for Theorem 1.4.

The problem considered in this article can be put under the umbrella of Calderón type inverse problems for parabolic partial differential equations (PDEs), which was initially proposed by Calderón in [14] for elliptic PDEs and studied by Nachman [44] in two dimensions and by Sylvester-Uhlmann in [54] in dimension three and higher. Analogous problems for parabolic and hyperbolic PDEs have been studied in [1, 31, 45, 48]. Choulli-Kian in [21] derived a stability estimate for recovering the time-dependent coefficient, which is a product of functions depending only on time and only space variables, from the boundary measurements. We also refer to [20], where an abstract inverse problem for parabolic pde is studied. All these works are concerned with the recovery of zeroth order perturbation of elliptic, parabolic, and hyperbolic PDEs from full boundary data. In [22], the recovery of general time-dependent zeroth order perturbation of heat operator from partial boundary measurements is considered. Inverse problems of recovering the coefficients appearing in the steady state convection-diffusion from full and partial boundary measurements in Euclidean geometry have been studied in [12, 17, 19, 24, 28, 38, 46, 47, 52]. Recovery of first-order perturbation of a parabolic pde from final and single measurement has been studied in [16] and [18] respectively. In [8], stable recovery of time-dependent coefficients appearing in a convection-diffusion from full boundary data has been studied. Choulli-Kian in [22] proved a stability estimate for recovering a time-dependent potential from partial boundary data, and motivated by their work, authors of [49] proved the unique recovery of time-dependent coefficients appearing in a convection-diffusion equation from partial boundary data. In [49], a uniqueness result is proved with a smallness assumption on the convection term, which is later on removed in a recent work of [51] where stability estimates for recovering the time-dependent coefficients of convection-diffusion equation from partial boundary data are derived. We also refer to [8] and [9] where stability estimates for convection-diffusion equation from full and partial boundary data is studied, respectively. Recently, in [15, 27], inverse problems related to nonlinear convection-diffusion equation is studied. In all the above-mentioned works, the inverse problems of recovering coefficients appearing in parabolic PDEs from full and partial boundary measurements in Euclidean geometry are considered. The inverse problems related to steady-state convection-diffusion equation in Riemannian geometry are considered in prior works (see, for example [11, 29, 32, 41]) however the recovery of time-dependent coefficients appearing in parabolic PDEs in Riemannian geometry has not been considered in prior works, and this is the main objective of this paper. To the best of our knowledge, this is the first work that considers the partial data inverse problem for recovering both first and zeroth order time-dependent perturbations of evolution equations in Riemannian geometry. Next, we mention works on inverse problems related to hyperbolic and dynamical Schrödinger equations, which are closely related to the study of this work. Inspired by [13] and [5, 6] authors of [33, 34, 35, 39, 42] studied the unique recovery of time-dependent coefficients appearing in a hyperbolic pde from partial boundary measurements. We also refer to [2, 3, 4, 7, 10, 26, 36, 37, 50] for inverse problems related to hyperbolic PDEs and dynamical Schrödinger equation in Euclidean as well as in Riemannian geometry, where time-dependent coefficients are recovered from full boundary measurements.

The rest of the article is organized as follows. In section 2, we derive the boundary and interior Carleman estimates which we will use in section 3 to construct the exponentially growing as well as decaying solutions. The main Theorem 1.4 of the article will be proved in section 4. Finally, we conclude the article with Acknowledgements.

2. Boundary and interior Carleman Estimates

The present section is devoted to prove boundary and interior Carleman estimates. The boundary Carleman estimate will be required to estimate the boundary terms in the integral identity where there is no measurement, and the interior Carleman estimate will be required to construct the geometric optics solutions (GO) for $\mathcal{L}_{A,q}$ and its formal L^2-adjoint. For a compact Riemannian manifold (M, g) with boundary denoted by ∂M, we denote by dV_g the volume form on (M, g) and by dS_g the induced volume form on ∂M. Then the L^2-norm of a function u on M and f on ∂M are given by

$$\|u\|_{L^2(M)} := \left(\int_M |u(x)|^2 \, dV_g\right)^{1/2} \text{ and } \|f\|_{L^2(\partial M)} := \left(\int_{\partial M} |f(x)|^2 \, dS_g\right)^{1/2}, \text{ respectively.}$$

We denote by $L^2(M)$ as the space of all functions u defined on M for which $\|u\|_{L^2(M)} < \infty$ and $L^2(\partial M)$ as the space of all functions f defined on ∂M for which $\|f\|_{L^2(\partial M)} < \infty$. Then $\left(L^2(M), \|\cdot\|_{L^2(M)}\right)$ and $\left(L^2(\partial M), \|\cdot\|_{L^2(\partial M)}\right)$ are Hilbert spaces with respect to the inner-products defined by

$$\langle f, g\rangle_{L^2(M)} := \int_M f(x)\overline{g(x)} \, dV_g$$

and

$$\langle f,g\rangle_{L^2(\partial M)} := \int_{\partial M} f(x)\overline{g(x)}\, dS_g$$

respectively.

We state the boundary Carleman estimate as follows.

Theorem 2.1. *Let (M,g) be an admissible manifold. For $\beta \in \left(\frac{1}{\sqrt{3}},1\right)$, let $\varphi(t,x) := \lambda^2\beta^2 t + \lambda x_1$, $A \in \left(W^{1,\infty}(M_T)\right)^n$, and $q \in L^\infty(M_T)$. Then there exists a constant $C>0$ depending only on M, T, A and q such that*

$$\begin{aligned}&\lambda^2\|e^{-\varphi}u\|^2_{L^2(M_T)} + \left\|e^{-\varphi}\nabla_g u\right\|^2_{L^2(M_T)} + \|e^{-\varphi(T,\cdot)}\nabla_g u(T,\cdot)\|^2_{L^2(M)} + \lambda\int_{\Sigma_+} e^{-2\varphi}|\partial_\nu u(t,x)|^2\nu_1\, dS_g dt\\ &\quad \le C\left(\|e^{-\varphi}\mathcal{L}_{A,q}u\|^2_{L^2(M_T)} + \lambda^2\|e^{-\varphi(T,\cdot)}u(T,\cdot)\|^2_{L^2(M)} + \lambda\int_{\Sigma_-} e^{-2\varphi}|\partial_\nu u(t,x)|^2|\nu_1|\, dS_g dt\right)\end{aligned} \tag{2.1}$$

hold for λ large enough and for all $u \in C^2(\overline{M_T})$ satisfying the following

$$u(0,x)=0,\ \text{for } x\in M \quad \text{and } u(t,x)=0,\ \text{for } (t,x)\in\Sigma.$$

The ν_1 appearing in (2.1) is given by $\nu_1 := \langle \nu, \frac{\partial}{\partial x_1}\rangle_g$.

Proof. In order to prove the weighted H^1-L^2 estimate given by (2.1), we need to convexify the Carleman weight φ. This convexification will help us to absorb the first-order perturbation A appearing in $\mathcal{L}_{A,q}$, which has been used in [15, 51] for Euclidean case and [29] for anisotropic magnetic Schrödinger operator. Now for $s>0$, we denote the convexified weight function by φ_s and define by

$$\varphi_s(t,x) := \varphi(t,x) - \frac{s(x_1+2\ell)^2}{2} = \lambda^2\beta^2 t + \lambda x_1 - \frac{s(x_1+2\ell)^2}{2}, \tag{2.2}$$

where ℓ is a positive real number such that x_1 is varying in $[-\ell,\ell]$ and the existence of such an ℓ is assured by the compactness assumption on M.
A direct computation gives

$$\partial_t\varphi_s = \lambda^2\beta^2,\ \ \partial_{x_1}\varphi_s = \lambda - s(x_1+2\ell),\ \ \partial^2_{x_1}\varphi_s = -s,\ \text{and } |\partial_{x_1}\varphi_s|^2 = (\lambda - s(x_1+2\ell))^2. \tag{2.3}$$

Before we proceed further, let us observe that

$$\mathcal{L}_{A,q}v(t,x) = \partial_t v(t,x) - \partial^2_{x_1}v(t,x) - \Delta_{g_0}v(t,x) - 2\langle A(t,x),\nabla_g v(t,x)\rangle_g + \tilde{q}(t,x)v(t,x)$$

where $\langle\cdot,\cdot\rangle_g$ and ∇_g denote the inner-product and gradient operator w.r.t. metric g respectively and $\tilde{q}(t,x) := q(t,x) - \delta_g A(t,x) - |A(t,x)|^2_g$, here in expression of $\tilde{q}$, $\delta_g A$ given by

$$\delta_g A = \frac{1}{\sqrt{|g|}}\sum_{j,k=1}^n \partial_j\left(g^{jk}\sqrt{|g|}A_k\right)$$

is known as the divergence operator w.r.t. to metric g and $|A|^2_g = \sum_{j,k=1}^n g^{jk}A_jA_k$. With this, we define the conjugated operator P_s with a convexified weight function φ_s by

$$P_s v := e^{-\varphi_s}\mathcal{L}_{A,q}\left(e^{\varphi_s}v\right) = e^{-\varphi_s}\left(\partial_t - \partial^2_{x_1} - \Delta_{g_0} - 2\langle A,\nabla_g\rangle_g + \tilde{q}\right)\left(e^{\varphi_s}v\right). \tag{2.4}$$

Upon expanding the above expression, P_s will take the following form

$$\begin{aligned}P_s v(t,x) &= \left(\partial_t + (\partial_t\varphi_s)\right)v(t,x) - \left(\partial^2_{x_1} + 2\partial_{x_1}\varphi_s\partial_{x_1} + (\partial_{x_1}\varphi_s)^2 + \partial^2_{x_1}\varphi_s\right)v - \Delta_{g_0}v(t,x)\\ &\quad - 2\langle A(t,x),\nabla_g v(t,x)\rangle_g - 2\langle A(t,x),\nabla_g\varphi_s(t,x)\rangle_g v(t,x) + \tilde{q}(t,x)v(t,x).\end{aligned}$$

Use relation (2.3) in the above equation to get

$$\begin{aligned} P_s v(t,x) &= \partial_t v(t,x) + \lambda^2\beta^2 v(t,x) - \left(\partial_{x_1}^2 + 2(\lambda - s(x_1+2\ell))\partial_{x_1} + (\lambda - s(x_1+2\ell))^2 - s\right) v(t,x) \\ &\quad - \Delta_{g_0} v(t,x) - 2\langle A(t,x), \nabla_g v(t,x)\rangle_g - 2\left(\lambda - s(x_1+2\ell)\right) g^{1k} A_k(t,x) v(t,x) + \tilde{q}(t,x) v(t,x). \end{aligned}$$

Now if we define P_1, P_2 and P_3 by

$$\begin{aligned} P_1 v(t,x) &:= \left(\partial_t v - 2(\lambda - s(x_1+2\ell))\partial_{x_1} v + 4sv\right)(t,x), \\ P_2 v(t,x) &:= \left(-\partial_{x_1}^2 v - \Delta_{g_0} v - \lambda^2(1-\beta^2)v + 2\lambda s(x_1+2\ell)v - s^2(x_1+2\ell)^2 v - 3sv\right)(t,x) \\ &:= \left(-\partial_{x_1}^2 - \Delta_{g_0} + \mathcal{K}(x_1)\right) v(t,x), \text{ where } \mathcal{K}(x_1) := 2\lambda s(x_1+2\ell) - \lambda^2(1-\beta^2) - s^2(x_1+2\ell)^2 - 3s \\ P_3 v(t,x) &:= -2\langle A(t,x), \nabla_g v(t,x)\rangle_g - 2\left(\lambda - s(x_1+2\ell)\right) g^{1k} A_k(t,x) v(t,x) + \tilde{q}(t,x) v(t,x) \end{aligned}$$

then one can check that $P_s v(t,x)$ has the following compact form

$$P_s v(t,x) = P_1 v(t,x) + P_2 v(t,x) + P_3 v(t,x). \tag{2.5}$$

Our first aim is to estimate the L^2 norm of $P_s v$ on M_T, therefore we define I_s by

$$\begin{aligned} I_s &:= \int_{M_T} |P_s v(t,x)|^2 \, dV_g(x)dt = \int_{M_T} |P_1 v(t,x) + P_2 v(t,x) + P_3 v(t,x)|^2 \, dV_g dt \\ &\geq \frac{1}{2}\int_{M_T} \left(P_1 v(t,x) + P_2 v(t,x)\right)^2 \, dV_g dt - \int_{M_T} |P_3 v(t,x)|^2 \, dV_g dt \\ &\geq \int_{M_T} P_1 v(t,x) P_2 v(t,x) \, dV_g dt - \int_{M_T} |P_3 v(t,x)|^2 \, dV_g dt. \end{aligned}$$

This gives us

$$I_s \geq \underbrace{\int_{M_T} P_1 v(t,x) P_2 v(t,x) \, dV_g dt}_{I_{s,1}} - \underbrace{\int_{M_T} |P_3 v(t,x)|^2 \, dV_g dt}_{I_{s,2}}. \tag{2.6}$$

We aim to estimate the right-hand side of (2.6). To do that, we start with the first term in the above inequality and, therefore consider

$$\begin{aligned} P_1 v(t,x) P_2 v(t,x) &= -\partial_t v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x) + \mathcal{K}(x_1) v(t,x)\partial_t v(t,x) - 4sv(t,x)\left(\partial_{x_1}^2 + \Delta_{g_0}\right) v(t,x) \\ &\quad + 4s\mathcal{K}(x_1)|v(t,x)|^2 + 2(\lambda - s(x_1+2\ell))\partial_{x_1} v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x) \\ &\quad - 2\mathcal{K}(x_1)(\lambda - s(x_1+2\ell)) v(t,x)\partial_{x_1} v(t,x). \end{aligned}$$

Now consider $I_{s,1}$ from (2.6)

$$\begin{aligned} I_{s,1} &= \int_{M_T} P_1 v(t,x) P_2 v(t,x) \, dV_g dt = -\int_{M_T} \partial_t v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x) \, dV_g dt \\ &\quad + 4s\int_{M_T} \mathcal{K}(x_1)|v(t,x)|^2 \, dV_g dt + \frac{1}{2}\int_{M_T} \mathcal{K}(x_1)\partial_t |v(t,x)|^2 \, dV_g dt \\ &\quad - 4s\int_{M_T} v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x) \, dV_g dt \\ &\quad + 2\int_{M_T} (\lambda - s(x_1+2\ell))\partial_{x_1} v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x) \, dV_g dt \\ &\quad - \int_{M_T} \mathcal{K}(x_1)(\lambda - s(x_1+2\ell))\partial_{x_1} |v(t,x)|^2 \, dV_g dt \\ &:= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \end{aligned}$$

where

$$I_1 := -\int_{M_T} \partial_t v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x)\, dV_g dt;\quad I_2 := 4s\int_{M_T} \mathcal{K}(x_1)|v(t,x)|^2\, dV_g dt$$
$$I_3 := \frac{1}{2}\int_{M_T} \mathcal{K}(x_1)\partial_t |v(t,x)|^2\, dV_g dt;\quad I_4 := -4s\int_{M_T} v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x)\, dV_g dt$$
$$I_5 := 2\int_{M_T} (\lambda - s(x_1+2\ell))\partial_{x_1} v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x)\, dV_g dt$$
$$I_6 := -\int_{M_T} \mathcal{K}(x_1)(\lambda - s(x_1+2\ell))\partial_{x_1}|v(t,x)|^2\, dV_g dt.$$

In order to estimate $I_{s,1}$ in (2.6), we need to estimate each I_j for $1 \le j \le 6$. To estimate these $I_j's$, we use integration by parts repeatedly along with initial and boundary conditions on v. Consider

$$I_1 = -\int_{M_T} \partial_t v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x)\, dV_g dt = \frac{1}{2}\int_M |\nabla_g v(T,x)|_g^2\, dV_g. \tag{2.7}$$

$$\begin{aligned} I_2 &= 4s\int_{M_T} \mathcal{K}(x_1)|v(t,x)|^2\, dV_g dt \\ &= -4s\int_{M_T} \left(\lambda^2(1-\beta^2) - 2\lambda s(x_1+2\ell) + s^2(x_1+2\ell)^2 + 3s\right)|v(t,x)|^2\, dV_g dt. \end{aligned} \tag{2.8}$$

$$\begin{aligned} I_3 &= \frac{1}{2}\int_{M_T} \mathcal{K}(x_1)\partial_t|v(t,x)|^2\, dV_g dt = \frac{1}{2}\int_M \mathcal{K}(x_1)|v(T,x)|^2 dV_g \\ &= \frac{1}{2}\int_M \left(-\lambda^2(1-\beta^2) + 2\lambda s(x_1+2\ell) - s^2(x_1+2\ell)^2 - 3s\right)|v(T,x)|^2\, dV_g. \end{aligned}$$

Recall $\ell \le (x_1+2\ell) \le 3\ell$, therefore choosing λ large enough, we obtain

$$I_3 \ge -C\lambda^2 \|v(T,\cdot)\|^2_{L^2(M)}, \quad \text{for some constant } C>0 \text{ independent of } \lambda. \tag{2.9}$$

Next, consider

$$I_4 = -4s\int_{M_T} v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x)\, dV_g dt = 4s\int_{M_T} |\nabla_g v(t,x)|_g^2\, dV_g dt. \tag{2.10}$$

The next integral in the line is

$$\begin{aligned} I_5 &= 2\int_{M_T} (\lambda - s(x_1+2\ell))\partial_{x_1} v(t,x)\left(\partial_{x_1}^2 v + \Delta_{g_0} v\right)(t,x)\, dV_g dt \\ &= 2\int_{M_T} (\lambda - s(x_1+2\ell))\partial_{x_1} v(t,x)\Delta_g v(t,x)\, dV_g dt. \end{aligned}$$

Using the integration by parts, we have

$$\begin{aligned} I_5 = &-2\int_{M_T} \partial_{x_1} v(t,x)\Big\langle \nabla_g v, \nabla_g\left(\lambda - s(x_1+2\ell)\right)\Big\rangle_g\, dV_g dt \\ &- 2\int_{M_T} (\lambda - s(x_1+2\ell))\Big\langle \nabla_g v, \partial_{x_1}\nabla_g v\Big\rangle_g\, dV_g dt \\ &+ 2\int_{\Sigma} (\lambda - s(x_1+2\ell))\partial_{x_1} v(t,x)\partial_\nu v(t,x)\, dS_g dt \end{aligned}$$

where $\nu(x)$ is outward unit normal vector to ∂M at $x \in \partial M$, $\partial_\nu v(t,x)$ stands for the normal derivative with respect t0 x of v at $(t,x) \in (0,T)\times\partial M$ and dS_g denotes the surface measure on ∂M. Again using the integration by parts, we have that

$$I_5 = 2s\int_{M_T} |\partial_{x_1} v(t,x)|^2\, dV_g dt - \int_\Sigma (\lambda - s(x_1+2\ell))\big\langle \nu, \frac{\partial}{\partial x_1}\big\rangle_g |\nabla_g v(t,x)|_g^2\, dS_g dt$$
$$- s\int_{M_T} |\nabla_g v(t,x)|_g^2\, dV_g dt + 2\int_\Sigma (\lambda - s(x_1+2\ell))\partial_{x_1} v(t,x)\partial_\nu v(t,x)\, dS_g dt.$$

Now $v|_{(0,T)\times\partial M} = 0$, implies that $\nabla_g v|_\Sigma = (\partial_\nu v)\,\nu$ and $\partial_{x_1} v|_\Sigma = \big\langle \nabla_g v, \frac{\partial}{\partial x_1}\big\rangle_g|_\Sigma = \partial_\nu v\big\langle \nu, \frac{\partial}{\partial x_1}\big\rangle_g = \partial_\nu v \nu_1$. Using these, we get

$$I_5 = s\int_{M_T} \left(|\partial_{x_1} v(t,x)|^2 - |\nabla_{g_0} v(t,x)|_{g_0}^2\right) dV_g dt + \int_\Sigma (\lambda - s(x_1+2\ell))|\partial_\nu v(t,x)|^2 \nu_1\, dS_g dt.$$

Combining I_5 and I_4, we have

$$\begin{aligned} I_4 + I_5 &= 5s\int_{M_T} |\partial_{x_1} v(t,x)|^2 dV_g dt + 3s\int_{M_T} |\nabla_{g_0} v(t,x)|_{g_0}^2 dV_g dt \\ &\quad + \int_\Sigma (\lambda - s(x_1+2\ell))|\partial_\nu v(t,x)|^2 \nu_1\, dS_g dt \\ &\geq 3s\,\|\nabla_g v\|_{L^2(M_T)}^2 + \int_\Sigma (\lambda - s(x_1+2\ell))|\partial_\nu v(t,x)|^2 \nu_1\, dS_g dt. \end{aligned} \tag{2.11}$$

Next, we consider the last term of $I_{s,1}$

$$\begin{aligned} I_6 &= -\int_{M_T} \mathcal{K}(x_1)(\lambda - s(x_1+2\ell))\partial_{x_1}|v(t,x)|^2\, dV_g dt \\ &= \int_{M_T} (\lambda - s(x_1+2\ell))\,|v(t,x)|^2 \partial_{x_1}\mathcal{K}(x_1)\, dV_g dt - s\int_{M_T} \mathcal{K}(x_1)|v(t,x)|^2\, dV_g dt \\ &= 2s\int_{M_T} (\lambda - s(x_1+2\ell))^2\,|v(t,x)|^2\, dV_g dt \\ &\quad + s\int_{M_T} \left(\lambda^2(1-\beta^2) - 2\lambda s(x_1+2\ell) + s^2(x_1+2\ell)^2 + 3s\right)|v(t,x)|^2\, dV_g dt. \end{aligned}$$

After simplifying, we get

$$\begin{aligned} I_6 &= s\lambda^2(3-\beta^2)\int_{M_T} |v(t,x)|^2\, dV_g dt - 6\lambda s^2\int_{M_T} (x_1+2\ell)|v(t,x)|^2\, dV_g dt \\ &\quad + 3s^2\int_{M_T} \left(s(x_1+2\ell)^2 + 1\right)|v(t,x)|^2\, dV_g dt. \end{aligned} \tag{2.12}$$

Next, we estimate $I_{s,2}$ in the following way:

$$\begin{aligned} I_{s,2} &= \int_{M_T} |P_3 v(t,x)|^2 dV_g dt \\ &= \int_{M_T} \Big|-2\langle A(t,x), \nabla_g v(t,x)\rangle_g - 2\,(\lambda - s(x_1+2\ell))\,g^{1k}A_k(t,x)v(t,x) + \tilde{q}(t,x)v(t,x)\Big|^2 dV_g dt \\ &\leq 8\|A\|_{L^\infty(M_T)}^2\|\nabla_g v\|_{L^2(M_T)}^2 + 8\lambda^2\|A\|_{L^\infty(M_T)}^2\|v\|_{L^2(M_T)}^2 + 2\|\tilde{q}\|_{L^\infty(M_T)}^2\|v\|_{L^2(M_T)}^2 \end{aligned} \tag{2.13}$$

Combining I_2, I_4, I_5, I_6, and $I_{s,2}$ in the following way:

$$I_2+I_4+I_5+I_6-I_{s,2}=\lambda^2\left(s\left(3\beta^2-1\right)-8\|A\|^2_{L^\infty(M_T)}-\frac{2}{\lambda^2}\|\tilde{q}\|^2_{L^\infty(M_T)}\right)\|v\|^2_{L^2(M_T)}$$
$$+\,(\text{terms having lower power of }\lambda)\,\|v\|^2_{L^2(M_T)}+\left(3s-8\|A\|^2_{L^\infty(M_T)}\right)\|\nabla_g v\|^2_{L^2(M_T)}$$
$$+\int_\Sigma(\lambda-s(x_1+2\ell))|\partial_\nu v|^2\nu_1\;dS_g dt.$$

After choosing s and λ large enough together with using the fact that $\beta\in(1/\sqrt{3},1)$, and a combination of all estimates obtained above, will amount to have the following estimate on $\|P_s v\|^2_{L^2(M_T)}$:

$$\|P_s v\|^2_{L^2(M_T)}\geq C\left(s\lambda^2\|v\|^2_{L^2(M_T)}+\|\nabla_g v(T,\cdot)\|^2_{L^2(M)}-\lambda^2\|v(T,\cdot)\|^2_{L^2(M)}+s\,\|\nabla_g v\|^2_{L^2(M_T)}\right.$$
$$\left.+\lambda\int_\Sigma|\partial_\nu v(t,x)|^2\nu_1\;dS_g dt\right) \tag{2.14}$$

where constant C depends only on A, q, T and M.

This provides the estimate for the operator $P_s=e^{-\varphi_s(t,x)}\mathcal{L}_{A,q}e^{\varphi_s(t,x)}$. To obtain the required Carleman estimate, we put $v(t,x)=e^{-\varphi_s(t,x)}u(t,x)$

$$\|e^{-\varphi_s}\mathcal{L}_{A,q}u\|^2_{L^2(M_T)}+\lambda^2 C\|e^{-\varphi_s(T,\cdot)}u(T,\cdot)\|^2_{L^2(M)}\;+\lambda\int_{\Sigma_-}|e^{-\varphi_s}\partial_\nu u(t,x)|^2|\nu_1|\;dS_g dt$$
$$\geq s\lambda^2\|e^{-\varphi_s}u\|^2_{L^2(M_T)}+\|e^{-\varphi_s(T,\cdot)}\nabla_g u(T,\cdot)\|^2_{L^2(M)}+s\left\|e^{-\varphi_s}\nabla_g u\right\|^2_{L^2(M_T)}$$
$$+\lambda\int_{\Sigma_+}|e^{-\varphi_s}\partial_\nu u(t,x)|^2\nu_1\;dS_g dt.$$

Finally, using the expression for $\varphi(t,x)$ and the fact that $e^{-\frac{s(x_1+2\ell)^2}{2}}$ has a strictly positive lower and upper bound, we get the following required estimate

$$\lambda^2\|e^{-\varphi}u\|^2_{L^2(M_T)}+\left\|e^{-\varphi}\nabla_g u\right\|^2_{L^2(M_T)}+\|e^{-\varphi(T,\cdot)}\nabla_g u(T,\cdot)\|^2_{L^2(M)}+\lambda\int_{\Sigma_+}e^{-2\varphi}|\partial_\nu u(t,x)|^2\nu_1\;dS_g dt$$
$$\leq C\left(\|e^{-\varphi}\mathcal{L}_{A,q}u\|^2_{L^2(M_T)}+\lambda^2\|e^{-\varphi(T,\cdot)}u(T,\cdot)\|^2_{L^2(M)}+\lambda\int_{\Sigma_-}e^{-2\varphi}|\partial_\nu u(t,x)|^2|\nu_1|\;dS_g dt\right)$$

for some constant $C>0$ independent of λ. This completes the proof of the theorem. □

Our next aim of this section is to derive the interior Carleman estimates in a Sobolev space of negative order for $\mathcal{L}_{A,q}$ and its formal L^2-adjoint $\mathcal{L}^*_{A,q}$. Before going to state and prove the interior Carleman estimates, we first give some definitions and notations for large parameter λ-dependent Sobolev spaces of arbitrary order. This will help us to represent the Carleman estimates in a nice form. Let us begin by assuming that (M,g) is embedded in a compact Riemannian manifold (N,g) without boundary and denote by $N_T:=(0,T)\times N$. Following [29], we denote by J^s for $s\in\mathbb{R}$, the large parameter λ-dependent pseudo-differential operator of order s on (N,g) and it is defined by $J^s:=\left(\lambda^2-\Delta_g\right)^{s/2}$. Using this, we define the large parameter λ-dependent Sobolev space $H^s_\lambda(N)$ for $s\in\mathbb{R}$, as the completion of $C^\infty(N)$ with respect to the following norm

$$\|u\|_{H^s_\lambda(N)}:=\|J^s u\|_{L^2(N)}.$$

Since (N,g) is a Riemannian manifold without boundary therefore the dual of $H^s_\lambda(N)$, for any $s\in\mathbb{R}$ can be identified with $H^{-s}_\lambda(N)$. Also note that for $s=1$, we have that

$$\|u\|^2_{H^1_\lambda(N)}:=\lambda^2\|u\|^2_{L^2(N)}+\|\nabla_g u\|^2_{L^2(N)}.$$

Now following [25] the time-dependent Sobolev spaces $L^2(0,T;H^s_\lambda(N))$ is defined as the set of all strongly measurable functions $u:[0,T]\to H^s_\lambda(N)$ such that

$$\|u\|_{L^2(0,T;H^s_\lambda(N)} := \left(\int_0^T \|u(t,\cdot)\|^2_{H^s_\lambda(N)}\, dt\right)^{1/2} < \infty. \tag{2.15}$$

Then $L^2(0,T;H^s_\lambda(N))$ is a Banach space with respect to the norm $\|\cdot\|_{L^2(0,T;H^s_\lambda(N))}$ defined by (2.15) and the dual of $L^2(0,T;H^s_\lambda(N))$ can be identified with $L^2(0,T;H^{-s}_\lambda(N))$. If we take $v\in C_c^\infty(M_T)$ in (2.14) then we have the following estimate

$$\|v\|^2_{L^2(0,T;H^1_\lambda(N))} \le C\|\mathcal{L}_\varphi v\|^2_{L^2(0,T;L^2(N))} \tag{2.16}$$

where φ is same as in Theorem 2.1 and $\mathcal{L}_\varphi := e^{-\varphi}\mathcal{L}_{A,q}e^{\varphi}$. Now if we denote by $\mathcal{L}^*_\varphi := e^{\varphi}\mathcal{L}^*_{A,q}e^{-\varphi}$ where $\mathcal{L}^*_{A,q}$ stands for a formal L^2-adjoint of $\mathcal{L}_{A,q}$ then using the arguments similar to the one used in deriving (2.14), the following estimate

$$\|u\|^2_{L^2(0,T;H^1_\lambda(N))} \le C\|\mathcal{L}^*_\varphi u\|^2_{L^2(0,T;L^2(N))} \tag{2.17}$$

holds for all $u\in C_c^\infty(M_T)$ where φ is same as in Theorem 2.1 and constant $C>0$ is independent of λ and u.

In order to construct the suitable solutions to $\mathcal{L}^*_{A,q}u=0$ and $\mathcal{L}_{A,q}v=0$, we need to shift the index by -1 for spacial variable in (2.16) and (2.17) respectively, which we will do in the following lemma.

Lemma 2.2. *Let $\mathcal{L}^*_\varphi := e^{\varphi}\mathcal{L}^*_{A,q}e^{-\varphi}$, and $\mathcal{L}_\varphi := e^{-\varphi}\mathcal{L}_{A,q}e^{\varphi}$, where $\mathcal{L}^*_{A,q}$ denote the formal L^2-adjoint of $\mathcal{L}_{A,q}$ and φ, A and q be as in Theorem 2.1. Then there exists a constant $C>0$ independent of λ and v such that*

$$\|v\|_{L^2([0,T];L^2(N))} \le C\|\mathcal{L}^*_\varphi v\|_{L^2(0,T;H^{-1}_\lambda(N))} \tag{2.18}$$

holds for all λ large enough and for all $v\in C_c^\infty(M_T)$ and

$$\|v\|_{L^2([0,T];L^2(N))} \le C\|\mathcal{L}_\varphi v\|_{L^2(0,T;H^{-1}_\lambda(N))} \tag{2.19}$$

holds for all λ large enough and for all $v\in C_c^\infty(M_T)$.

Proof. First, we establish (2.18), and the proof for (2.19) can be carried out in a similar manner. We begin with the inequality:

$$\|v\|^2_{L^2(0,T;H^1_\lambda(N))} \le C\|\mathcal{L}^*_\varphi v\|^2_{L^2(0,T;L^2(N))}$$

holds for all $v\in C_c^\infty(M_T)$. Next, we shift the index by -1 in the above estimate. Let $w\in C_c^\infty(M_T)$ and consider the adjoint operator defined as:

$$\mathcal{L}^*_{A,q} := \Big(-\partial_t - \sum_{j,k=1}^n \frac{1}{\sqrt{|g|}}(\partial_{x_j}-A_j)(\sqrt{|g|}g^{jk}(\partial_{x_k}-A_k)) + \overline{q}\Big).$$

For $s>0$, define the convexified weight function φ_s as follows:

$$\varphi_s(t,x) := \varphi(t,x) + \frac{s(x_1+2\ell)^2}{2} = \lambda^2\beta^2 t + \lambda x_1 + \frac{s(x_1+2\ell)^2}{2}.$$

Let $P^*_s := e^{\varphi_s}\mathcal{L}^*_{A,q}e^{-\varphi_s}$, we have

$$P^*_s w = e^{\varphi_s}\left(-\partial_t - \partial^2_{x_1} - \Delta_{g_0} + 2\langle A,\nabla_g\rangle_g + \tilde{q}^*\right)(e^{-\varphi_s}w)$$

where $\tilde{q}^*(t,x) := \overline{q}(t,x) + \delta_g A(t,x) - |A(t,x)|^2_g$.

Expressing $P^*_s w$ as a sum of three components:

$$P^*_s w := P^*_1 w(t,x) + P^*_2 w(t,x) + P^*_3 w(t,x)$$

where

$$P_1^* w(t,x) := \left(-\partial_t w + 2(\lambda + s(x_1+2\ell))\partial_{x_1} w + 4sw\right)(t,x) := -\partial_t w(t,x) + \widetilde{P_1}^* w(t,x)$$
$$P_2^* w(t,x) := \left(-\partial_{x_1}^2 w - \Delta_{g_0} w - \lambda^2(1-\beta^2)w - 2\lambda s(x_1+2\ell)w - s^2(x_1+2\ell)^2 w - 3sw\right)(t,x),$$
$$P_3^* w(t,x) := 2\langle A(t,x), \nabla_g w(t,x)\rangle_g - 2\left(\lambda + s(x_1+2\ell)\right) g^{1k} A_k(t,x) w(t,x) + \tilde{q}^*(t,x) w(t,x).$$

where $\widetilde{P_1}^* w := 2\left(\lambda + s(x_1+2\ell)\right)\partial_{x_1} w + 4sw$. Now if we denote the symbols of the pseudo-differential operators $\widetilde{P_1}^*, P_2^*, J$ and J^{-1} by $\widetilde{p_1}^*, p_2^*, j$ and j^{-1} respectively then they are given by

$$\widetilde{p_1}^*(x,\xi) = 2\left(\lambda + s(x_1+2\ell)\right)\xi_1 + 4s,\ \ p_2^*(x,\xi) = |\xi|_g^2 - \lambda^2(1-\beta^2) - 2\lambda s(x_1+2\ell) - s^2(x_1+2\ell)^2 - 3s$$
$$j(\xi,\lambda) = \left(\lambda^2 + |\xi|_g^2\right)^{1/2} \text{ and } j^{-1}(\xi,\lambda) = \left(\lambda^2 + |\xi|_g^2\right)^{-1/2}$$

where $|\xi|_g^2 := \xi_1^2 + |\xi'|_{g_0}^2$ stands for the symbol of $-\Delta_g := -\partial_{x_1}^2 - \Delta_{g_0}$ (see [55, Page 353]). Now since both J and J^{-1} commute with ∂_t appearing in P_1^* therefore using the properties of the composition of pseudo-differential operators (see [30, Theorem 18.1.8] and [15, Proposition 4.1]), we have

$$J^{-1}(P_1^* + P_2^*)J^1 w = (P_1^* + P_2^*)w + E_\lambda w$$

where E_λ is the pseudo-differential operator of order 1 with symbol given by

$$E_\lambda(x,\xi) = \frac{i\xi_1}{\left(\lambda^2 + |\xi|_g^2\right)^{3/2}}\left(2is\xi_1 - 2\lambda s - 2s^2(x_1+2\ell)\right)\left(\lambda^2 + |\xi|_g^2\right)^{1/2} + \underset{\left(\lambda^2+|\xi|_g^2\right)^{1/2}\to\infty}{o}(1) \tag{2.20}$$

Also note that while deriving (2.20), we have used the fact that $\widetilde{P_1}^* + P_2^*$ depends only on x_1 and now using (2.20) along with the properties of pseudo-differential operators, we have

$$\|E_\lambda w\|_{L^2(0,T;L^2(N))} \le Cs^2 \|w\|_{L^2(0,T;L^2(N))} \tag{2.21}$$

holds for all $C_c^\infty(M_T)$. Combining the above estimates along with the triangle inequality, we get

$$\begin{aligned}\|(P_1^* + P_2^*)J^1 w\|^2_{L^2(0,T;H_\lambda^{-1}(N))} &= \|J^{-1}(P_1^* + P_2^*)J^1 w\|^2_{L^2(0,T;L^2(N))}\\ &\ge \frac{1}{2}\|(P_1^* + P_2^*)w\|^2_{L^2(0,T;L^2(N))} - \|E_\lambda w\|^2_{L^2(0,T;L^2(N))}.\end{aligned}$$

Following the same calculations as done for the Carleman estimate (2.1) together with the estimate (2.21) and taking $\frac{\lambda^2}{s^3}$ sufficiently large, we obtain

$$\|(P_1^* + P_2^*)J^1 w\|^2_{L^2(0,T;H_\lambda^{-1}(N))} \ge C'\left(s\|\nabla_g w\|^2_{L^2(0,T;L^2(N))} + s\lambda^2\|w\|^2_{L^2(0,T;L^2(N))}\right) - Cs^4\|w\|^2_{L^2(0,T;L^2(N))} \tag{2.22}$$
$$\ge C''\left(s\|\nabla_g w\|^2_{L^2(0,T;L^2(N))} + s\lambda^2\|w\|^2_{L^2(0,T;L^2(N))}\right). \tag{2.23}$$

Also using the expression for P_3^* given above and boundedness of A and q, we obtain the following estimate

$$\|P_3^* J^1 w\|^2_{L^2(0,T;H_\lambda^{-1}(N))} \le C(\|A\|_\infty^2\|\nabla_g w\|^2_{L^2(0,T;L^2(N))} + \lambda^2\|A\|_\infty^2\|w\|^2_{L^2(0,T;L^2(N))} + \|\tilde{q}^*\|_\infty^2\|w\|^2_{L^2(0,T;L^2(N))}) \tag{2.24}$$

Hence using (2.22) and (2.24) and choosing s and λ large enough, we get

$$\|P_s^* J^1 w\|_{L^2(0,T;H_\lambda^{-1}(N))} \ge C\|w\|_{L^2(0,T;H_\lambda^1(N))}.$$

Now, consider $\chi \in C_c^\infty(\widetilde{M})$ such that $\chi = 1$ in $\overline{M_1}$ where $\overline{M} \subset M_1 \subset \widetilde{M}$. By taking $w = \chi J^{-1} v$ in the above estimate and using the estimates

$$\|(1-\chi)J^{-1}v\|_{L^2(0,T;H_\lambda^1(N))} \le \frac{C}{\lambda^2}\|v\|_{L^2(0,T;L^2(N))}$$

and

$$\|v\|_{L^2(0,T;L^2(N))} = \|J^{-1}v\|_{L^2(0,T;H^1_\lambda(N))} \leq \|w\|_{L^2(0,T;H^1_\lambda(N))} + \frac{C}{\lambda^2}\|v\|_{L^2(0,T;L^2(N))}$$

we get

$$\begin{aligned}\|P_s^* v\|_{L^2(0,T;H^{-1}_\lambda(N))} &\geq \|P_s^* J^1 w\|_{L^2(0,T;H^{-1}_\lambda(N))} - \frac{C}{\lambda^2}\|v\|_{L^2(0,T;L^2(N))} \\ &\geq \|w\|_{L^2(0,T;H^1_\lambda(N))} - \frac{C}{\lambda^2}\|v\|_{L^2(0,T;L^2(N))} \\ &\geq C\|v\|_{L^2(0,T;L^2(N))}\end{aligned}$$

holds for λ large. Now, using the expression for $\varphi(t,x)$ and the fact that $e^{\frac{s(x_1+2\ell)^2}{2}}$ has a strictly positive lower and upper bound, we conclude that

$$\|v\|_{L^2([0,T];L^2(N))} \leq C\|\mathcal{L}_\varphi^* v\|_{L^2(0,T;H^{-1}_\lambda(N))}$$

holds for all λ large enough and for all $v \in C_c^\infty(M_T)$. This completes the proof of (2.18). □

The above estimates, together with the Hahn-Banach theorem and the Riesz representation theorem, give the following solvability result, proof of which follows from [49, 51].

Lemma 2.3. *Let φ, A and q be as before and $\lambda > 0$ be large enough. Then for $F \in L^2(M_T)$ there exists a solution $u \in H^1(0,T;H^{-1}(M)) \cap L^2(0,T;H^1(M))$ of*

$$\mathcal{L}_\varphi w(t,x) = F(t,x),\ (t,x) \in M_T$$

satisfying the following estimate

$$\|u\|_{L^2(0,T;H^1_\lambda(M))} \leq C\|F\|_{L^2(M_T)} \tag{2.25}$$

for some constant $C > 0$ independent of λ and u and there exists a solution $v \in H^1(0,T;H^{-1}(M)) \cap L^2(0,T;H^1(M))$ of

$$\mathcal{L}_\varphi^* w(t,x) = F(t,x),\ (t,x) \in M_T$$

satisfying the following estimate

$$\|v\|_{L^2(0,T;H^1_\lambda(M))} \leq C\|F\|_{L^2(M_T)} \tag{2.26}$$

for some constant $C > 0$ independent of λ and v.

Proof. The proof for $\mathcal{L}_\varphi$ is presented below, and the proof for $\mathcal{L}_\varphi^*$ can be established using analogous arguments.
Consider the subspace S of $L^2(0,T;H^{-1}_\lambda(N))$ defined as

$$S := \{\mathcal{L}_\varphi^* w(t,x) : w \in C_c^\infty(M_T)\}.$$

Define the linear operator T on S by

$$T(\mathcal{L}_\varphi^* z) = \int_{M_T} z(t,x)\overline{F(t,x)}\ dV_g dt,\ \text{for } F \in L^2(M_T).$$

For any $\mathcal{L}_\varphi^* z \in S$, we have

$$|T(\mathcal{L}_\varphi^* z)| \leq \int_{M_T} |z(t,x)||F(t,x)|\ dV_g dt \leq \|z\|_{L^2(M_T)}\|F\|_{L^2(M_T)}.$$

Using the Carleman estimate (2.18), we obtain

$$|T(\mathcal{L}_\varphi^* z)| \leq C\|F\|_{L^2(M_T)}\|\mathcal{L}_\varphi^* z\|_{L^2(0,T;H^{-1}_\lambda(N))}.$$

This inequality holds for $z \in C_c^\infty(M_T)$. By the Hahn-Banach theorem, extend the linear operator T to $L^2(0,T;H_\lambda^{-1}(N))$. Denote the extended map as T and note that it satisfies the inequality

$$\|T\| \le C\|F\|_{L^2(M_T)}.$$

By the Riesz representation theorem, as T is a bounded linear functional on $L^2(0,T;H_\lambda^{-1}(N))$, there exists a unique $u \in L^2(0,T;H_\lambda^1(N))$ such that

$$T(f) = \langle f, u\rangle_{L^2(0,T;H_\lambda^{-1}(N)),L^2(0,T;H_\lambda^1(N))} \text{ for } f \in L^2(0,T;H_\lambda^{-1}(N)),$$

with

$$\|u\|_{L^2(0,T;H_\lambda^1(N))} \le C\|F\|_{L^2(M_T)}.$$

Now, for $z \in C_c^\infty(M_T)$, choosing $f = \mathcal{L}_\varphi^* z$ in the above equation, we get $\mathcal{L}_\varphi u = F$.
Using the expression for $\mathcal{L}_\varphi$ and the fact that $u \in L^2(0,T;H^1(M))$ and $F \in L^2(M_T)$, we conclude that $\partial_t u \in L^2(0,T;H^{-1}(M))$. Hence, we have $u \in H^1(0,T;H^{-1}(M)) \cap L^2(0,T;H^1(M))$.

□

3. Construction of geometric optics solutions

In this section, we aim to construct the exponential growing and decaying solutions to the convection-diffusion operator $\mathcal{L}_{A,q}$ and its L^2-adjoint $\mathcal{L}_{A,q}^*$, respectively. Construction of these solutions will be proved with the help of the interior Carleman estimates in negative order Sobolev spaces stated in Lemma 2.2.

3.1. **Construction of exponentially growing solutions.** In this subsection, we will construct the exponential growing solutions to $\mathcal{L}_{A,q}u(t,x) = 0$, in M_T which take the following form

$$u(t,x) = e^{(\varphi+i\psi)(t,x)}\Big(T_g(t,x) + R_{g,\lambda}(t,x)\Big) \tag{3.1}$$

where φ is the same as in Theorem 2.1 and ψ, T_g will be constructed using the WKB construction in such a way that the correction term $R_{g,\lambda}$ satisfies the following

$$e^{-(\varphi+i\psi)}\mathcal{L}_{A,q}\left(e^{(\varphi+i\psi)}R_{g,\lambda}(t,x)\right) = F_\lambda(t,x),\ (t,x) \in M_T$$

for $F_\lambda \in L^2(M_T)$ such that $\|F_\lambda\|_{L^2(M_T)} \le C$, for some constant $C > 0$ independent of λ and $R_{g,\lambda}$ satisfies $\|R_{g,\lambda}\|_{L^2(0,T;H_\lambda^1(M))} \le C\|F_\lambda\|_{L^2(M_T)}$, for some constant $C > 0$, not depending on λ. More precisely, we prove the following theorem.

Theorem 3.1. *Let M_T, $\mathcal{L}_{A,q}$ and φ be as before. Let (D, g_0) be a simple manifold satisfying $M_0 \subset D$ and there exists a $y_0 \in D$ such that $(x_1, y_0) \notin M$ for all x_1. Now if (r,θ) denote the polar normal coordinates on (D, g_0), (x_1, r, θ) denote the points in M and A_1 and A_r are components of A in x_1 and r coordinates respectively, then for λ large enough the following equation*

$$\mathcal{L}_{A,q}v(t,x) = 0, \quad (t,x) \in M_T$$

has a solution taking the following form

$$u(t,x) = e^{\varphi+i\psi}\Big(T_g(t,x_1,r,\theta) + R_{g,\lambda}(t,x_1,r,\theta)\Big) \tag{3.2}$$

where

$$\psi = \lambda(\sqrt{1-\beta^2})r, \text{ and } T_g(t,x_1,r,\theta) = \phi(t)e^{i\mu\left(\sqrt{1-\beta^2}\right)x_1}e^{-\mu r}e^{i\Phi_1(t,x_1,r,\theta)}b(r,\theta)^{-1/4}h(\theta)$$

here $\phi \in C_c^\infty(0,T)$, μ is a real number, Φ_1 is solution to

$$\partial_1\Phi_1 + i(\sqrt{1-\beta^2})\partial_r\Phi_1 + \Big(-iA_1 + (\sqrt{1-\beta^2})A_r\Big) = 0$$

and $R_{g,\lambda}$ satisfies the following

$$\mathcal{L}_{\varphi}\left(e^{i\psi}R_{g,\lambda}\right)(t,x) = -e^{i\psi}\mathcal{L}_{A,q}T_g(t,x), \quad (t,x)\in M_T$$

and $\|R_{g,\lambda}\|_{L^2\left(0,T;H^1_\lambda(M)\right)} \leq C$ for some constant $C>0$ independent of λ.

Proof. Following [29], if we denote $\rho := \varphi + i\psi$, then simple calculations show that the conjugated operator $\mathcal{L}_\rho := e^{-\rho}\mathcal{L}_{A,q}e^{\rho}$ will have the following expression

$$\mathcal{L}_\rho = \mathcal{L}_{A,q} + \left(\partial_t\rho - \Delta_g\rho - g^{jk}\partial_j\rho\partial_k\rho\right) - 2\left(g^{jk}\partial_j\rho\partial_k + g^{jk}\partial_j\rho A_k\right).$$

Using $\rho = \varphi + i\psi$, and $\varphi = \lambda^2\beta^2 t + \lambda x_1$, we get

$$\begin{aligned}\mathcal{L}_\rho = \mathcal{L}_{A,q} &+ \left(\lambda^2\beta^2 - \lambda^2 + g^{jk}\partial_j\psi\partial_k\psi\right)\\ &- \left(2\lambda\partial_1 + 2ig^{jk}\partial_j\psi\partial_k + 2\lambda A_1 + 2ig^{jk}\partial_j\psi A_k + i\Delta_g\psi + 2i\lambda\partial_1\psi - i\partial_t\psi\right).\end{aligned} \tag{3.3}$$

Now u given by (3.1) solves $\mathcal{L}_{A,q}v = 0$ if and only if $\mathcal{L}_\rho\left(e^{-\rho}u\right) = 0$. This will give us

$$\begin{aligned}\mathcal{L}_\rho R_{g,\lambda}(t,x) = -\mathcal{L}_{A,q}T_g(t,x) &- \left(\lambda^2\beta^2 - \lambda^2 + g^{jk}\partial_j\psi\partial_k\psi\right)T_g(t,x)\\ &+ \left(2\lambda\partial_1 + 2ig^{jk}\partial_j\psi\partial_k + 2\lambda A_1 + 2ig^{jk}\partial_j\psi A_k + i\Delta_g\psi + 2i\lambda\partial_1\psi - i\partial_t\psi\right)T_g(t,x), \quad (t,x)\in M_T.\end{aligned} \tag{3.4}$$

In order to have $\|R_{g,\lambda}\|_{L^2(0,T;H^1_\lambda(M)} \leq C$, we choose ψ and T_g such that

$$\partial_t\psi = 0, \quad g^{jk}\partial_j\psi\partial_k\psi = \lambda^2(1-\beta^2) \tag{3.5}$$

and

$$\left(2\lambda\partial_1 + 2ig^{jk}\partial_j\psi\partial_k + 2\lambda A_1 + 2ig^{jk}\partial_j\psi A_k + i\Delta_g\psi + 2i\lambda\partial_1\psi\right)T_g(t,x) = 0, \ (t,x)\in M_T. \tag{3.6}$$

To solve equations (3.5) and (3.6) for ψ and T_g, we use the polar normal coordinates (r,θ) on (D,g_0) centered at $y_0\in D$ as mentioned in statement of theorem. We consider the polar normal coordinates on D which are denoted by (r,θ) and given by $x_0 = \exp_{y_0}(r\theta)$, where $r>0$ and $\theta\in S_{y_0}(D) := \{v\in T_{y_0}D : |v|_g = 1\}$, here $T_{y_0}D$ denote the tangent space to D at $y_0\in D$. Then using the Gauss lemma (see Lemma 15 in Chapter 9 of [53]) there exists a smooth positive definite matrix $P(r,\theta)$ with $\det(P) := b(r,\theta)$ such that the metric g_0 in the polar normal coordinates (r,θ), takes the following form

$$g_0(r,\theta) = \begin{bmatrix}1 & 0\\ 0 & P(r,\theta)\end{bmatrix}. \tag{3.7}$$

Now since the points in M are denoted by (x_1,r,θ) where (r,θ) are polar normal coordinates in (D,g_0), therefore after using the previous form of g_0, the metric g has the following form

$$g(x_1,r,\theta) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & P(r,\theta)\end{bmatrix}. \tag{3.8}$$

Using (3.8), we see that

$$\psi(x) = (\lambda\sqrt{1-\beta^2})\mathrm{dist}_g(y_0,x) = (\lambda\sqrt{1-\beta^2})r, \tag{3.9}$$

solves equation (3.5) on M. Using this choice of ψ and form of g given by (3.8) in equation (3.6), we have

$$\left(\partial_1 + i(\sqrt{1-\beta^2})\partial_r + A_1 + i(\sqrt{1-\beta^2})A_r + i(\sqrt{1-\beta^2})\frac{\partial_r b(r,\theta)}{4b(r,\theta)}\right)T_g(t,x_1,r,\theta) = 0.$$

Now, one can check that the solution of the above equation can be given by

$$T_g(t,x_1,r,\theta)=\phi(t)e^{i\mu\left(\sqrt{1-\beta^2}\right)x_1}e^{-\mu r}e^{i\Phi_1(t,x_1,r,\theta)}b(r,\theta)^{-1/4}h(\theta) \tag{3.10}$$

where $\phi\in C_c^\infty(0,T)$, $\mu\in\mathbb{R}$, $h\in C^\infty(S_{y_0}(D))$ are arbitrary but fixed and $\Phi_1(t,x_1,r,\theta)$ satisfies the following

$$\left(\partial_1\Phi_1+i\left(\sqrt{1-\beta^2}\right)\partial_r\Phi_1\right)+\left(-iA_1+(\sqrt{1-\beta^2})A_r\right)=0. \tag{3.11}$$

Now using (3.9) and (3.10) in (3.4), we get

$$\mathcal{L}_\rho R_{g,\lambda}(t,x)=-\mathcal{L}_{A,q}T_g(t,x),\quad (t,x)\in M_T$$

But $\mathcal{L}_\rho R_{g,\lambda}=e^{-i\psi}\mathcal{L}_\varphi\left(e^{i\psi}R_{g,\lambda}\right)$ therefore if we denote $\widetilde{R}_{g,\lambda}=e^{i\psi}R_{g,\lambda}$ then $\widetilde{R}_{g,\lambda}$ satisfies the following equation

$$\mathcal{L}_\varphi\widetilde{R}_{g,\lambda}(t,x)=-e^{i\psi}\mathcal{L}_{A,q}T_g(t,x),\quad (t,x)\in M_T \tag{3.12}$$

Now using the expressions for ψ and T_g from (3.9) and (3.10) respectively and assumptions on A and q, we have that $-e^{i\psi}\mathcal{L}_{A,q}T_g\in L^2(M_T)$ and $\|e^{i\psi}\mathcal{L}_{A,q}T_g\|_{L^2(M_T)}\leq C$, for some constant $C>0$ independent of λ. Hence using Lemma 2.3 together with above estimate for right hand side of (3.12), we conclude that there exists $\widetilde{R}_{g,\lambda}\in H^1(0,T;H^{-1}(M))\cap L^2(0,T;H^1(M))$ solving (3.12) and it satisfies the following estimate $\|\widetilde{R}_{g,\lambda}\|_{L^2(0,T;H^1_\lambda(M))}\leq C$, for some constant $C>0$, independent of λ. Hence, we conclude that $R_{g,\lambda}$ solves the required equation and satisfies the desired estimate. This completes the proof of the Theorem. □

3.2. **Construction of exponentially decaying solutions.** The aim of this subsection is to construct the exponential decaying solutions to

$$\mathcal{L}^*_{A,q}u:=\Big(-\partial_t-\sum_{j,k=1}^n\frac{1}{\sqrt{|g|}}(\partial_{x_j}-A_j)(\sqrt{|g|}g^{jk}(\partial_{x_k}-A_k))+\overline{q}\Big)u=0,\ \text{in } M_T$$

taking the following form

$$u(t,x)=e^{-(\varphi-i\psi)(t,x)}\Big(T_d(t,x)+R_{d,\lambda}(t,x)\Big) \tag{3.13}$$

where φ is the same as in Theorem 2.1 and ψ, T_d will be constructed using the WKB construction in such a way that the correction term $R_{d,\lambda}$ satisfies the following

$$e^{(\varphi-i\psi)}\mathcal{L}^*_{A,q}\left(e^{-(\varphi-i\psi)}R_{d,\lambda}(t,x)\right)=F_\lambda(t,x),\ (t,x)\in M_T$$

for some $F_\lambda\in L^2(M_T)$ such that $\|F_\lambda\|_{L^2(M_T)}\leq C$, for some constant $C>0$ independent of λ and $R_{d,\lambda}$ satisfies $\|R_{d,\lambda}\|_{L^2(0,T;H^1_\lambda(M))}\leq C\|F_\lambda\|_{L^2(M_T)}$, for some constant $C>0$ not depending on λ. To construct these solutions, we first start with the construction of ψ and T_d following the arguments used in Theorem 3.1. Denote by $\rho:=\varphi-i\psi$, then one can check that the conjugated operator

$$\mathcal{L}^*_\rho:=e^\rho\mathcal{L}^*_{A,q}e^{-\rho}=e^\rho\Big(-\partial_t-\sum_{j,k=1}^n\frac{1}{\sqrt{|g|}}(\partial_{x_j}-A_j)(\sqrt{|g|}g^{jk}(\partial_{x_k}-A_k))+\overline{q}\Big)e^{-\rho}$$

is given by

$$\mathcal{L}^*_\rho=\mathcal{L}^*_{A,q}+\Big(\partial_t\rho-g^{jk}\partial_j\rho\partial_k\rho\Big)+\Big(2g^{jk}\partial_j\rho\partial_k-2g^{jk}\partial_j\rho A_k+\Delta_g\rho\Big).$$

Using $\rho=\varphi-i\psi$ and $\varphi=\lambda^2\beta^2t+\lambda x_1$, we have

$$\mathcal{L}^*_\rho=\mathcal{L}^*_{A,q}+\Big(\lambda^2\beta^2-\lambda^2+g^{jk}\partial_j\psi\partial_k\psi\Big)+\Big(2\lambda\partial_1-2ig^{jk}\partial_j\psi\partial_k-2\lambda A_1+2ig^{jk}\partial_j\psi A_k-i\Delta_g\psi+2i\lambda\partial_1\psi-i\partial_t\psi\Big).$$

Now we observe that u given by (3.13) solves $\mathcal{L}^*_{A,q}v = 0$ in M_T if and only if $\mathcal{L}^*_\rho(e^\rho u) = 0$ in M_T. Using this, we see that $R_{d,\lambda}$ satisfies the following equation

$$\begin{aligned}\mathcal{L}^*_\rho R_{d,\lambda}(t,x) = &-\mathcal{L}^*_{A,q}T_d(t,x) - \left(\lambda^2\beta^2 - \lambda^2 + g^{jk}\partial_j\psi\partial_k\psi\right)T_d(t,x)\\ &- \left(2\lambda\partial_1 - 2ig^{jk}\partial_j\psi\partial_k - 2\lambda A_1 + 2ig^{jk}\partial_j\psi A_k - i\Delta_g\psi + 2i\lambda\partial_1\psi - i\partial_t\psi\right)T_d(t,x).\end{aligned} \tag{3.14}$$

To get the estimate $\|R_{d,\lambda}\|_{L^2(0,T;H^1_\lambda(M))} \leq C$, for some constant $C > 0$ independent of λ, we choose ψ and T_d satisfying the following equations

$$\partial_t\psi = 0, \quad \lambda^2\beta^2 - \lambda^2 + g^{jk}\partial_j\psi\partial_k\psi = 0 \tag{3.15}$$

and

$$\left(2\lambda\partial_1 - 2ig^{jk}\partial_j\psi\partial_k - 2\lambda A_1 + 2ig^{jk}\partial_j\psi A_k - i\Delta_g\psi + 2i\lambda\partial_1\psi\right)T_d(t,x) = 0, \ (t,x) \in M_T \tag{3.16}$$

respectively. To solve equations (3.15) and (3.16) for ψ and T_d, we again use the polar normal coordinates (r,θ) on (D,g_0) centered at $y_0 \in D$ as used in the proof of Theorem 3.1. For a fixed $y_0 \in D$, we consider the polar normal coordinates on D which are denoted by (r,θ) and given by $x_0 = \exp_{y_0}(r\theta)$, where $r > 0$ and $\theta \in S_{y_0}(D) : \{v \in T_{y_0}D : \ |v|_g = 1\}$, here $T_{y_0}D$ denote the tangent space to D at $y_0 \in D$. Then using the Gauss lemma (see Lemma 15 in Chapter 9 of [53]) there exists a smooth positive definite matrix $P(r,\theta)$ with $\det P(r,\theta) = b(r,\theta)$ such that the metric g_0 in the polar normal coordinates (r,θ), takes form given by (3.7). Now since the points in M are denoted by (x_1,r,θ) where (r,θ) are polar normal coordinates in (D,g_0), therefore after using the form of g_0 given by (3.7), the metric g takes the form given by equation (3.8) and using this, we observe that

$$\psi(x) = \left(\lambda\sqrt{1-\beta^2}\right)\mathrm{dist}_g(y_0,x) = \left(\lambda\sqrt{1-\beta^2}\right)r, \tag{3.17}$$

solves equation (3.15) and

$$T_d(t,x_1,r,\theta) = \phi(t)e^{i\Phi_2(t,x_1,r,\theta)}b(r,\theta)^{-1/4}h(\theta) \tag{3.18}$$

solves equation (3.16) where $\phi \in C^\infty_c(0,T)$, $h \in C^\infty(S_{y_0}(D))$ are arbitrary but fixed and $\Phi_2(t,x_1,r,\theta)$ satisfies the following

$$\left(\partial_1\Phi_2 - i\left(\sqrt{1-\beta^2}\right)\partial_r\Phi_2\right) + \left(iA_1 + \sqrt{1-\beta^2}A_r\right) = 0, \tag{3.19}$$

A_1 and A_r are components of A in x_1 and r coordinates respectively. Now if we use (3.17) and (3.18) in equation (3.14) and repeating the arguments used in showing the estimate for $R_{g,\lambda}$ in Theorem 3.1, then we get that there exists $R_{d,\lambda} \in H^1(0,T;H^{-1}(M)) \cap L^2(0,T;H^1(M))$ solving

$$\mathcal{L}^*_\varphi\left(e^{i\psi}R_{d,\lambda}\right)(t,x) = -e^{i\psi}\mathcal{L}_{A,q}T_d(t,x), \quad (t,x) \in M_T \tag{3.20}$$

and $R_{d,\lambda}$ satisfies the following estimate

$$\|R_{d,\lambda}\|_{L^2(0,T;H^1_\lambda(M)} \leq C \tag{3.21}$$

for some constant $C > 0$ independent of λ. Combining all these, we end up with proving the following theorem.

Theorem 3.2. *Let M_T, $\mathcal{L}_{A,q}$ and φ be as before. Let (D,g_0) be a simple manifold which is extension of (M_0,g_0) in the sense that $M_0 \subset D$ and there exists a $y_0 \in D$ such that $(x_1,y_0) \notin M$ for all x_1. Now if (r,θ) denote the polar normal coordinates on (D,g_0), (x_1,r,θ) denote the points in M and A_1 and A_r are components of A in x_1 and r coordinates respectively, then for λ large enough the following equation*

$$\mathcal{L}^*_{A,q}v(t,x) = 0, \quad (t,x) \in M_T$$

has a solution taking the following form

$$v(t,x) = e^{-(\varphi - i\psi)(t,x)}\Big(T_d(t,x_1,r,\theta) + R_{d,\lambda}(t,x_1,r,\theta)\Big) \tag{3.22}$$

where ψ, T_d *are given by* (3.17), (3.18) *and* $R_{d,\lambda}$ *satisfies* (3.20) *and* (3.21).

4. Derivation of Integral identity and proof of Main Theorem

We use this section to derive an integral identity, which will be required to prove our main result. Later, using the geometric optics solutions constructed in Section 3, we conclude the proof of Theorem 1.4. We start by recalling

$$\mathcal{L}_{A,q} = \partial_t - \sum_{j,k=1}^{n} \frac{1}{\sqrt{|g|}}\left(\partial_{x_j} + A_j(t,x)\right)\left(g^{jk}\sqrt{|g|}(\partial_{x_k} + A_k(t,x))\right) + q(t,x)$$

and

$$\mathcal{L}^*_{A,q} = -\partial_t - \sum_{j,k=1}^{n} \frac{1}{\sqrt{|g|}}(\partial_{x_j} - A_j(t,x))\left(\sqrt{|g|}g^{jk}(\partial_{x_k} - A_k(t,x))\right) + \overline{q}(t,x).$$

For $l = 1,2$, let $A^{(l)}$ and q_l be as in Theorem 1.4. Further assume that u_l is solution to the corresponding IBVP for $\mathcal{L}_{A^{(l)},q_l}$ given by (1.1) when $(A,q) = (A^{(l)}, q_l)$ for $l = 1,2$, that is, for $l = 1,2$, we have

$$\begin{cases} \mathcal{L}_{A^{(l)},q_l} u_l(t,x) = 0, & (t,x) \in M_T \\ u_l(0,x) = \phi(x), & x \in M \\ u_l(t,x) = f(t,x), & (t,x) \in \Sigma. \end{cases} \tag{4.1}$$

Then $u := u_1 - u_2$, satisfies the following IBVP with zero initial and boundary conditions

$$\begin{cases} \mathcal{L}_{A^{(1)},q_1} u(t,x) = \mathcal{Q}u_2(t,x), & (t,x) \in M_T \\ u(0,x) = 0, & x \in M \\ u(t,x) = 0, & (t,x) \in \Sigma, \end{cases} \tag{4.2}$$

where $\mathcal{Q}u_2(t,x) := \left(|A^{(1)}|_g^2 - |A^{(2)}|_g^2\right)u_2 + 2\left\langle A^{(1)} - A^{(2)}, \nabla_g u_2\right\rangle_g + \delta_g\left(A^{(1)} - A^{(2)}\right)u_2 + (q_2 - q_1)u_2$. To simplify the notation, let us denote by $\tilde{q}(t,x) := (\tilde{q}_1 - \tilde{q}_2)(t,x)$ and $\tilde{A}(t,x) := (\tilde{A})_{1\le j\le n} := (A^{(1)} - A^{(2)})(t,x)$ where $\tilde{q}_i := |A^{(i)}|_g^2 + \delta_g A^{(i)} - q_i$, for $i = 1,2$, then with these notations $\mathcal{Q}u_2$ becomes

$$\mathcal{Q}u_2(t,x) = 2\langle \tilde{A}(t,x), \nabla_g u_2(t,x)\rangle_g + \tilde{q}(t,x)u_2(t,x).$$

Now since $\mathcal{Q}u_2 \in L^2(M_T)$ therefore using Theorem 1.43 in [23] we have that there exists a unique solution $u \in L^2(0,T;H^2(M)) \cap H^1(0,T;L^2(M))$ to (4.2) with $\partial_\nu u \in L^2(0,T;H^{1/2}(\Sigma))$. Now if $v(t,x)$ is a solution to the adjoint operator of $\mathcal{L}_{A^{(1)},q_1}$, given by

$$\mathcal{L}^*_{A^{(1)},q_1} v(t,x) = 0, \quad (t,x) \in M_T, \tag{4.3}$$

then we observe that

$$
\begin{aligned}
&\langle(\Lambda_{A^{(1)},q_1}-\Lambda_{A^{(2)},q_2})(\phi,f),v|_{\partial M_T^*}\rangle=\langle\mathcal{N}_{A^{(1)},q_1}u_1-\mathcal{N}_{A^{(2)},q_2}u_2,v|_{\partial M_T^*}\rangle\\
&\quad=\int_{M_T}\Big(-u_1\partial_t\overline{v}+\langle\nabla_g u_1,\nabla_g\overline{v}\rangle_g+2u_1\langle A^{(1)},\nabla_g\overline{v}\rangle_g+(\delta_g A^{(1)})u_1\overline{v}-|A^{(1)}|_g^2u_1\overline{v}+q_1u_1\overline{v}\Big)\ dV_gdt\\
&\qquad-\int_M u_1(0,x)\bar{v}(0,x)dV_g\\
&\quad-\int_{M_T}\Big(-u_2\partial_t\overline{v}+\langle\nabla_g u_2,\nabla_g\overline{v}\rangle_g+2u_2\langle A^{(2)},\nabla_g\overline{v}\rangle_g+(\delta_g A^{(2)})u_2\overline{v}-|A^{(2)}|_g^2u_2\overline{v}+q_2u_2\overline{v}\Big)\ dV_gdt\\
&\qquad+\int_M u_2(0,x)\bar{v}(0,x)dV_g.
\end{aligned}
$$

Using integration by parts with $u|_{\Sigma}=0, u|_{t=0}=0$ and v is solution to (4.3), we get

$$
\langle(\Lambda_{A^{(1)},q_1}-\Lambda_{A^{(2)},q_2})(\phi,f),v|_{\partial M_T^*}\rangle=-2\int_{M_T}\langle\tilde{A}(t,x),\nabla_g u_2(t,x)\rangle_g\bar{v}(t,x)\ dV_gdt-\int_{M_T}\tilde{q}(t,x)u_2(t,x)\bar{v}(t,x)\ dV_gdt. \tag{4.4}
$$

Multiplying equation (4.2) by $\bar{v}(t,x)$ and integrate it over M_T, we get

$$
\int_{M_T}\mathcal{L}_{A^{(1)},q_1}u(t,x)\bar{v}(t,x)\ dV_gdt=2\int_{M_T}\langle\tilde{A}(t,x),\nabla_g u_2(t,x)\rangle_g\bar{v}(t,x)\ dV_gdt+\int_{M_T}\tilde{q}(t,x)u_2(t,x)\bar{v}(t,x)\ dV_gdt.
$$

Now use the integration by parts together with $u|_{\Sigma}=0$, $u|_{t=0}=0$, $\tilde{A}|_{\Sigma}=0$ and the fact that v is a solution to (4.3), to obtain the following identity

$$
\begin{aligned}
2\int_{M_T}&\langle\tilde{A}(t,x),\nabla_g u_2(t,x)\rangle_g\bar{v}(t,x)\ dV_gdt+\int_{M_T}\tilde{q}(t,x)u_2(t,x)\bar{v}(t,x)\ dV_gdt\\
&=-\int_{\Sigma}g^{jk}\nu_j\partial_{x_k}u(t,x)\bar{v}(t,x)\ dS_gdt+\int_M u(T,x)\bar{v}(T,x)\ dV_g.
\end{aligned} \tag{4.5}
$$

From Equations (4.4) and (4.5), we have

$$
\langle(\Lambda_{A^{(1)},q_1}-\Lambda_{A^{(2)},q_2})(\phi,f),v|_{\partial M_T^*}\rangle=\int_{\Sigma}g^{jk}\nu_j\partial_{x_k}u(t,x)\bar{v}(t,x)\ dS_gdt-\int_M u(T,x)\bar{v}(T,x)\ dV_g.
$$

Using (1.8), we get $\partial_\nu u|_{\Sigma_{-,\epsilon/2}}=0$ and $u|_{t=T}=0$. Therefore, Equation (4.5) becomes

$$
\begin{aligned}
2\int_{M_T}&\langle\tilde{A}(t,x),\nabla_g u_2(t,x)\rangle_g\bar{v}(t,x)\ dV_gdt+\int_{M_T}\tilde{q}(t,x)u_2(t,x)\bar{v}(t,x)\ dV_gdt\\
&=-\int_{\Sigma\setminus\Sigma_{-,\epsilon/2}}g^{jk}\nu_j\partial_{x_k}u(t,x)\bar{v}(t,x)\ dS_gdt.
\end{aligned} \tag{4.6}
$$

Let us define J_1, J_2 and J_3 by

$$
J_1:=2\int_{M_T}\langle\tilde{A}(t,x),\nabla_g u_2(t,x)\rangle_g\bar{v}(t,x)\ dV_gdt,\ \ J_2:=\int_{M_T}\tilde{q}(t,x)u_2(t,x)\bar{v}(t,x)\ dV_gdt\ \text{ and}
$$

$$
J_3:=-\int_{\Sigma\setminus\Sigma_{-,\epsilon/2}}\partial_\nu u(t,x)\bar{v}(t,x)\ dS_gdt.
$$

With these notations (4.6) becomes

$$
J_1+J_2=J_3. \tag{4.7}
$$

Our next aim is to substitute the exponentially growing and decaying solutions constructed in section 3, for u_2 and v respectively, in each term of equation (4.7). Recall that u_2 satisfies

$$
\mathcal{L}_{A^{(2)},q_2}u_2=0,\ \text{ in } M_T
$$

and v satisfies

$$\mathcal{L}^*_{A^{(1)},q_1} v = 0, \text{ in } M_T$$

therefore we choose the expressions for solutions u_2 and v from (3.2) and (3.22) respectively, substitute in each term of (4.7). We start with the following calculations

$$\begin{aligned}
\Big\langle \tilde{A}(t,x), \nabla_g u_2(t,x)\Big\rangle_g \bar{v} &= \Big(\Big\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi)\Big\rangle_g T_g(t,x) + \Big\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi)\Big\rangle_g R_{g,\lambda}(t,x) \\
&\quad + \Big\langle \tilde{A}, \nabla_g T_g(t,x)\Big\rangle_g + \Big\langle \tilde{A}, \nabla_g R_{g,\lambda}(t,x)\Big\rangle_g\Big)\Big(\bar{T}_d(t,x) + \bar{R}_{d,\lambda}(t,x)\Big). \\
&= \Big\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi)\Big\rangle_g T_g(t,x)\bar{T}_d(t,x) + \Big\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi)\Big\rangle_g T_g(t,x)\bar{R}_{d,\lambda}(t,x) \\
&\quad + \Big\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi)\Big\rangle_g R_{g,\lambda}(t,x)\bar{T}_d(t,x) + \Big\langle \tilde{A}(t,x)\nabla_g (\varphi + i\psi)\Big\rangle_g R_{g,\lambda}(t,x)\bar{R}_{d,\lambda}(t,x) \\
&\quad + \Big\langle \tilde{A}, \nabla_g T_g(t,x)\Big\rangle_g \bar{T}_d(t,x) + \Big\langle \tilde{A}, \nabla_g T_g(t,x)\Big\rangle_g \bar{R}_{d,\lambda}(t,x) \\
&\quad + \Big\langle \tilde{A}, \nabla_g R_{g,\lambda}(t,x)\Big\rangle_g \bar{T}_d(t,x) + \Big\langle \tilde{A}, \nabla_g R_{g,\lambda}(t,x)\Big\rangle_g \bar{R}_{d,\lambda}(t,x) \\
&:= \Big\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi)\Big\rangle_g T_g(t,x)\bar{T}_d(t,x) + \mathcal{Z}_1(t,x).
\end{aligned}$$

Similarly, we see that

$$\tilde{q}(t,x)u_2(t,x)\bar{v}(t,x) = \tilde{q}(t,x)\big(\bar{T}_d T_g(t,x) + \bar{T}_d R_{g,\lambda}(t,x) + T_g \bar{R}_{d,\lambda}(t,x) + \bar{R}_{d,\lambda}(t,x)R_{g,\lambda}(t,x)\big) := \mathcal{Z}_2(t,x).$$

Using the above expressions in definitions of J_1 and J_2, we get

$$\begin{aligned}
J_1 + J_2 = 2\int_{M_T} &\Big\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi)\Big\rangle_g T_g(t,x)\bar{T}_d(t,x)\ dV_g dt \\
&+ 2\int_{M_T} \mathcal{Z}_1(t,x)\ dV_g dt + \int_{M_T} \mathcal{Z}_2(t,x)\ dV_g dt.
\end{aligned} \tag{4.8}$$

Now using the expression for v from (3.22) in the expression of J_3, we obtain

$$J_3 = -\int_{\Sigma\setminus\Sigma_{-,\epsilon/2}} e^{-(\varphi+i\psi)}\partial_\nu u(t,x)\bar{T}_d(t,x)\ dS_g dt - \int_{\Sigma\setminus\Sigma_{-,\epsilon/2}} e^{-(\varphi+i\psi)}\partial_\nu u(t,x)\bar{R}_{d,\lambda}(t,x)\ dS_g dt.$$

We use the boundary Carleman estimate given in Theorem 2.1 and follow the arguments used in deriving Lemma 5.1 in [49] to get the following estimate for J_3

$$|J_3| \le C\lambda^{1/2}, \text{ for some constant } C > 0, \text{ independent of } \lambda. \tag{4.9}$$

Using (4.8) together with the estimate on J_3 given by (4.9) in (4.7), we get

$$\begin{aligned}
\left|2\int_{M_T} \Big\langle \tilde{A}(t,x), \nabla_g (\varphi + i\psi)\Big\rangle_g T_g(t,x)\bar{T}_d(t,x)\ dV_g dt\right| &\le \left|2\int_{M_T} \mathcal{Z}_1(t,x)\ dV_g dt + \int_{M_T} \mathcal{Z}_2(t,x)\ dV_g dt\right| + |J_3| \\
&\le C\left(\|\mathcal{Z}_1\|_{L^2(M_T)} + \|\mathcal{Z}_2\|_{L^2(M_T)} + |J_3|\right).
\end{aligned}$$

Let (x_1, r, θ) be the polar normal coordinate on (M,g) and $\tilde{A}_1$ and $\tilde{A}_r$ be components of $\tilde{A}$ in x_1 and r coordinates respectively as in Theorem 3.1. Then, the above estimate can be rewritten as

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{S_y M_0}\int_0^{\tau_+(y_0,\theta)} &\Big(\tilde{A}_1 + i\sqrt{1-\beta^2}\tilde{A}_r\Big) T_g(t,x_1,r,\theta)\bar{T}_d(t,x_1,r,\theta)b(r,\theta)^{1/2} dr d\theta dx_1 dt \\
&\le \frac{C}{\lambda}\left(\|\mathcal{Z}_1\|_{L^2(M_T)} + \|\mathcal{Z}_2\|_{L^2(M_T)} + |J_3|\right)
\end{aligned} \tag{4.10}$$

where we used the fact $dV_g = b(r,\theta)^{1/2}dx_1drd\theta$ in polar normal coordinates on (M,g) and $\tau_+(y_0,\theta)$ is length of the geodesic in M, starting at y_0 in the direction of θ. After using the estimates on $R_{d,\lambda}$ and $R_{g,\lambda}$ together with the expressions for T_d and T_g from Theorems 3.1 and 3.2, we get that

$$\|\mathcal{Z}_i\|_{L^2(M_T)} \leq C, \text{ for } i = 1,2 \text{ and constant } C > 0 \text{ independent of } \lambda.$$

Using this estimate along with equation (4.9) in equation (4.10) and taking $\lambda \to \infty$, we get

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{S_yM_0}\int_0^{\tau_+(y_0,\theta)} \left(\tilde{A}_1 + i\sqrt{1-\beta^2}\tilde{A}_r\right)(\phi(t))^2 e^{i\mu\left(\sqrt{1-\beta^2}\right)x_1} e^{-\mu r} e^{i\Phi(t,x_1,r,\theta)} (h(\theta))^2 \ drd\theta dx_1 dt = 0,$$

where $\Phi(t,x_1,r,\theta) := \left(\Phi_1 - \overline{\Phi}_2\right)(t,x_1,r,\theta)$ with Φ_1 and Φ_2 satisfying equations (3.11) and (3.19) respectively. As this relation is true for all cutoff functions $\phi \in C_c^\infty(0,T)$ therefore we get

$$\int_{\mathbb{R}}\int_{S_yM_0}\int_0^{\tau_+(y_0,\theta)} \left(\tilde{A}_1 + i\sqrt{1-\beta^2}\tilde{A}_r\right)(t,x_1,r,\theta) e^{i\mu\left(\sqrt{1-\beta^2}\right)x_1} e^{-\mu r} e^{i\Phi(t,x_1,r,\theta)} (h(\theta))^2 \ drd\theta dx_1 = 0,$$

for all $t \in (0,T)$ and $h \in C^\infty(S_{y_0}(D))$. Next, we vary $h \in C^\infty(S_{y_0}(D))$ in the above equation to get

$$\int_{\mathbb{R}}\int_0^{\tau_+(y_0,\theta)} \left(\tilde{A}_1 + i\sqrt{1-\beta^2}\tilde{A}_r\right)(t,x_1,r,\theta) e^{i\mu\left(\sqrt{1-\beta^2}\right)x_1} e^{-\mu r} e^{i\Phi(t,x_1,r,\theta)} \ drdx_1 = 0 \tag{4.11}$$

for all $t \in (0,T)$ and for all θ.

From here on, uniqueness of the convection term can be obtained by following exactly the same arguments used in [29]. However, we prefer to give a sketch of the proof for the sake of completeness; please refer [29, Section 6] for a detailed discussion.

Let $\rho = (\sqrt{1-\beta^2})x_1 + ir$, then $\Phi(t,x_1,r,\theta) := \left(\Phi_1 - \overline{\Phi}_2\right)(t,x_1,r,\theta)$ satisfies the following equation:

$$\frac{\partial \Phi}{\partial \overline{\rho}} + \frac{i}{2\sqrt{1-\beta^2}}\left(\tilde{A}_1 + i\sqrt{1-\beta^2}\tilde{A}_r\right) = 0. \tag{4.12}$$

For fixed θ, define $\Omega_\theta = \{(x_1,r) \in \mathbb{R}^2 : (x_1,r,\theta) \in M\}$. Then the identity (4.11) can be rewritten as follows:

$$\int\int_{\Omega_\theta} 2i\sqrt{1-\beta^2}\frac{\partial \Phi}{\partial \overline{\rho}} e^{i\mu\rho} e^{i\Phi} \ d\overline{\rho} \wedge d\rho = 0,$$

which reduces to

$$\int_{\partial\Omega_\theta} e^{i\Phi} e^{i\mu\rho} \ d\rho = 0.$$

Now following [28, 29], we find that there exists a non-vanishing holomorphic function $F \in C(\overline{\Omega_\theta})$ such that $e^{i\Phi}|_{\partial\Omega_\theta} = F|_{\partial\Omega_\theta}$ and its holomorphic logarithm satisfies (please see [28, Lemma 5.1]for more details)

$$\log F = i\Phi \quad \text{on} \quad \partial\Omega_\theta.$$

Since $\log F$ is a holomorphic function on Ω_θ, we get

$$\int_{\partial\Omega_\theta} i\Phi e^{i\mu\rho} \ d\rho = 0.$$

From Stokes' formula, we have

$$\int\int_{\Omega_\theta} i\frac{\partial \Phi}{\partial \overline{\rho}} e^{i\mu\rho} \ d\overline{\rho} \wedge d\rho = 0.$$

Using equation (4.12), we obtain the following identity

$$\int\int_{\Omega_\theta} \left(\tilde{A}_1 + i\sqrt{1-\beta^2}\tilde{A}_r\right) e^{i\mu\left(\sqrt{1-\beta^2}x_1 + ir\right)} \ d\overline{\rho} \wedge d\rho = 0, \quad \text{for all } \theta \text{ and for all } t.$$

Let $\gamma_{y_0,\theta}$ be the geodesic starting from y_0 and in the direction θ. The above identity can be rewritten as

$$\int e^{-\mu r}\left[f(\gamma_{y_0,\theta}(r))+i\sqrt{1-\beta^2}\alpha(\dot{\gamma}_{y_0,\theta}(r))\right]\,dr=0,$$

with

$$f(x')=\int e^{i\mu\sqrt{1-\beta^2}x_1}\tilde{A}_1(x_1,x')\,dx_1 \quad \text{and} \quad \alpha(x')=\sum_{j=2}^{n}\underbrace{\left(\int e^{i\mu\sqrt{1-\beta^2}x_1}\tilde{A}_j(x_1,x')\,dx_1\right)}_{\alpha_j}\,dx^j.$$

By varying y_0 on ∂D in Theorem 3.1 and using [29, Theorem 7.1], we obtain $f=-\mu p$ (for μ small) and $\alpha=\frac{-i}{\sqrt{1-\beta^2}}dp$ for $p\in C^\infty(D)$ with $p|_{\partial D}=0$. With these notations, we can simplify α as follows:

$$\alpha(x')=\sum_{j=2}^{n}\mathcal{F}\left(\tilde{A}_j\right)(\mu\sqrt{1-\beta^2},x')dx^j=\sum_{j=2}^{n}\frac{-i}{\sqrt{1-\beta^2}}\left(\partial_j p\right)dx^j.$$

In the above expression, $\mathcal{F}(\cdot)$ represents the Fourier transform with respect to x_1. The analyticity of the Fourier transform gives the following relation

$$\partial_k\tilde{A}_j-\partial_j\tilde{A}_k=0,\quad j,k=2,\dots,n.$$

For $2\le j\le n$, we have

$$-\mu\partial_j p=\partial_j f=\int e^{i\mu\sqrt{1-\beta^2}x_1}\partial_j\tilde{A}_1(x_1,x')\,dx_1,\quad \text{using the definition of } f$$

$$\partial_1\alpha_j=i\mu\sqrt{1-\beta^2}\alpha_j+\int e^{i\mu\sqrt{1-\beta^2}x_1}\partial_1\tilde{A}_j(x_1,x')\,dx_1=0,\quad \text{as } \alpha_j \text{ depends on } x' \text{ only.}$$

Use $i\mu\sqrt{1-\beta^2}\alpha_j=\mu\partial_j p$ in the above relation to get

$$\int e^{i\mu\sqrt{1-\beta^2}x_1}\left(\partial_j\tilde{A}_1-\partial_1\tilde{A}_j\right)(x_1,x')\,dx_1=0.$$

Thus, we have $d\tilde{A}=0$ in M_T, and consequently, we obtain that there exists a $\Psi\in W_0^{2,\infty}(M_T)$ such that $\tilde{A}(t,x)=\nabla_g\Psi(t,x)$ for $(t,x)\in M_T$. This proves the required uniqueness for the convection term. Next, we prove the uniqueness of density coefficient q. To prove this, we replace the pair $(A^{(1)},q_1)$ by $(A^{(3)},q_3)$, by taking $A^{(3)}=A^{(2)}$ in M_T, where $A^{(3)}(t,x)=A^{(1)}(t,x)-\nabla_g\Psi(t,x)$ and $q_3(t,x)=q_1(t,x)-\partial_t\Psi(t,x)$. From Proposition 1.3 and Equation (1.8), we get $\Lambda_{A^{(3)},q_3}=\Lambda_{A^{(2)},q_2}$. Using this in Equation (4.6), we get

$$\int_{M_T}(q_2-q_3)(t,x)u_2(t,x)\bar{v}(t,x)\,dV_g dt=-\int_{\Sigma\setminus\Sigma_{-,\epsilon/2}}g^{jk}\nu_j\partial_{x_k}u(t,x)\bar{v}(t,x)\,dS_g dt,$$

Again, we use the explicit expressions of u_2 and v from Theorems 3.1 and 3.2 and take $\lambda\to\infty$ together with the estimate $\|\mathcal{Z}_2\|_{L^2(M_T)}\le C/\lambda$, for some constant $C>0$, independent of λ, to end up with getting

$$\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{S_yM_0}\int_0^{\tau_+(y_0,\theta)}q(t,x_1,r,\theta)\,(\phi(t))^2\,e^{i\mu\left(\sqrt{1-\beta^2}\right)x_1}e^{-\mu r}e^{i\Phi(t,x_1,r,\theta)}\,(h(\theta))^2\,dt dx_1 dr d\theta=0,$$

where $q(t,x_1,r,\theta):=(q_2-q_3)(t,x_1,r,\theta)$ is assumed to be zero outside M_T. Finally by varying $\phi\in C_c^\infty(0,T)$, $h\in C^\infty(S_{y_0}(D))$ and taking $\Phi\equiv 0$ which is possible since $A^{(3)}=A^{(2)}$ in M_T and if Φ_1 solves (3.11) then we can choose $\Phi_2=\overline{\Phi}_1$ which solves (3.19), we get that

$$\int_{\mathbb{R}}\int_0^{\tau_+(y_0,\theta)}q(t,x_1,r,\theta)e^{i\mu\left(\sqrt{1-\beta^2}x_1+ir\right)}dx_1 dr=0,\ \text{for all } \theta\in S_{y_0}(D),\ \beta\in\left(\frac{1}{\sqrt{3}},1\right) \text{ and } t\in(0,T).$$

Now, using the arguments from [29, Section 6], we can prove the required uniqueness of density coefficient. Here, we provide a brief outline of the proof for completeness. We rewrite the above equation as

$$\int_0^\infty e^{-\mu r} \underbrace{\left(\int_{\mathbb{R}} e^{i\mu\sqrt{1-\beta^2}x_1} q(t,x_1,r,\theta)\ dx_1\right)}_{f(\gamma_{y_0,\theta}(r))} dr = 0$$

$$\implies \qquad \int_0^\infty f(\gamma_{y_0,\theta}(r)) \exp\left[-\int_0^r \mu\ ds\right]\ dr = 0$$

for all geodesics $\gamma_{y_0,\theta}$ starting from the point y_0. By varying y_0 such that $(x_1, y_0) \notin M$ for all x_1 and using the injectivity of the attenuated geodesic ray transform (with attenuation $-\mu$) from [29, Theorem 7.1], we get the following for sufficiently small μ

$$\int_{\mathbb{R}} e^{i\mu\sqrt{1-\beta^2}x_1} q(t,x_1,r,\theta)\ dx_1 = 0, \quad \text{for all } r, \theta \text{ and } t.$$

Since the above identity is the Fourier transform of q with respect to x_1; therefore, we have $q \equiv 0$ (that is, $q_2 \equiv q_3$) in M_T. Recall $q_3 = q_1(t,x) - \partial_t \Psi(t,x)$, hence, we obtain $q_1(t,x) - q_2(t,x) = \partial_t\Psi(t,x)$ for $(t,x) \in M_T$, which completes the proof of the main theorem.

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Declarations. Conflict of interest
On behalf of all authors, the corresponding author declare no conflict of interest.

References

[1] S.A. Avdonin and T.I. Seidman; *Identification of $q(x)$ in $u_t = \Delta u - qu$, from boundary observations,* SIAM J. Control Optim., **33** (1995), 1247-1255.

[2] I.B. Aïcha; *Stability estimate for an inverse problem for the Schrödinger equation in a magnetic field with time-dependent coefficient*, J. Math. Phys. **58** (2017), no. 7, 071508, 21 pp.

[3] M. Bellassoued and D. Dos Santos Ferreira; *Stable determination of coefficients in the dynamical anisotropic Schrödinger equation from the Dirichlet-to-Neumann map*, Inverse Problems, **26** (2010), no. 12, 125010, 30pp.

[4] M. Bellassoued and D. Dos Santos Ferreira; *Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map*, Inverse Probl. Imaging, **5** (2011), no. 4, 745–773.

[5] M. Bellassoued, D. Jellali and M. Yamamoto; Lipschitz stability for a hyperbolic inverse problem by finite local boundary data, *Appl. Anal.*, **85** (2006), no. 10, 1219–1243.

[6] M. Bellassoued, D. Jellali and M. Yamamoto; Stability estimate for the hyperbolic inverse boundary value problem by local Dirichlet-to-Neumann map, *J. Math. Anal. Appl.*, **343** (2008), no. 2, 1036–1046.

[7] M. Bellassoued, I.B. Aïcha and Z. Rezig; *Stable determination of a vector field in a non-self-adjoint dynamical Schrödinger equation on Riemannian manifolds*, Math. Control Relat. Fields, Vol. **11** (2021), no. 2, 403–431.

[8] M. Bellassoued and I. Rassas; *Stability estimate for an inverse problem of the convection-diffusion equation*, J. Inverse Ill-Posed Probl. Vol. **28** (2020), no. 1, 71–92.

[9] M. Bellassoued and O.B. Fraj; *Stably determining time-dependent convection-diffusion coefficients from a partial Dirichlet-to-Neumann map*, Inverse Problems, Vol. **37** (2021), no. 4, Paper No. 045011, 35 pp.

[10] M. Bellassoued and Z. Rezig; *Simultaneous determination of two coefficients in the Riemannian hyperbolic equation from boundary measurements*, Ann. Global Anal. Geom. Vol. **56** (2019), no. 2, 291–325.

[11] S. Bhattacharyya; *An inverse problem for the magnetic Schrödinger operator on Riemannian manifolds from partial boundary data*, Inverse Probl. Imaging **12** (2018), 801-830.

[12] M. R. Brown and M. Salo; *Identifiability at the boundary for first-order terms*, Appl. Anal. Vol. **85** (2006), no. 6-7, 735–749.

[13] A.L. Bukhgeim and G. Uhlmann; *Recovering a potential from partial Cauchy data*, Comm. Partial Differential Equations **27** (2002), no. 3-4, 653–668.

[14] A.P. Calderón; *On an inverse boundary value problem*, Seminar on Numerical Analysis and its Applications to Continuum Physics, Soc. Brasileira de Matematica, Rio de Janerio, 1980.

[15] P. Caro and Y. Kian; *Determination of convection terms and quasi-linearities appearing in diffusion equations*, preprint, arXiv1812.08495.

[16] Z. Cha Deng, J. Ning Yu, and Y. Liu; *Identifying the coefficient of first-order in parabolic equation from final measurement data*, Mathematics and Computers in Simulation, **77** (2008) 421-435.

[17] J. Cheng and M. Yamamoto; *The global uniqueness for determining two convection coefficients from Dirichlet to Neumann map in two dimensions*, Inverse Problems, **16** (2000), no. 3, L25–L30.

[18] J. Cheng and M. Yamamoto; *Identification of convection term in a parabolic equation with a single measurement*, Nonlinear Analysis, **50** (2002), 163-171.

[19] J. Cheng and M. Yamamoto; *Determination of Two Convection Coefficients from Dirichlet to Neumann Map in the Two-Dimensional Case*, SIAM J. Math. Anal., **35** (2004), 1371-1393.

[20] M. Choulli; *Abstract inverse problem and application*, J. Math. Anal. Appl. Vol. **160** (1991), no. 1, 190–202.

[21] M. Choulli and Y. Kian; *Stability of the determination of a time-dependent coefficient in parabolic equations*, Math. Control Relat. Fields Vol. **3** (2013), no. 2, 143–160.

[22] M. Choulli and Y. Kian; *Logarithmic stability in determining the time-dependent zero-order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map. Application to the determination of a nonlinear term*, J. Math. Pures Appl., **(9) 114** (2018), 235–261.

[23] M. Choulli, Une introduction aux problmes inverses elliptiques et paraboliques, Mathmatiques et Applications, Vol. 65, Springer-Verlag, Berlin, 2009.

[24] M. Cristofol and E. Soccorsi; *Stability estimate in an inverse problem for non-autonomous magnetic Schrödinger equations*, *Appl. Anal.*, **90**, 2011, no. 10, 1499–1520, 0003-6811.

[25] L.C. Evans; *Partial differential equations*, Second edition, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010. xxii+749 pp.

[26] A. Feizmohammadi, J. Ilmavirta, Y. Kian and L. Oksanen; *Recovery of time-dependent coefficients from boundary data for hyperbolic equations*, J. Spectr. Theory **11** (2021), no. 3, 1107–1143.

[27] A. Feizmohammadi, Y. Kian and G. Uhlmann; *An inverse problem for a quasilinear convection-diffusion equation*, Nonlinear Anal. Vol. **222** (2022), Paper No. 112921, 30 pp.

[28] D. Dos Santos Ferreira, C. E. Kenig, J. Sjöstrand and G. Uhlmann; *Determining a magnetic Schrödinger operator from partial Cauchy data*, Comm. Math. Phys., **271** (2007), no. 2, 467–488.

[29] D. D. S. Ferreira, C. E. Kenig, M. Salo and G. Uhlmann; *Limiting carleman weights and anisotropic inverse problems*, Invent. Math., **178** (2009), 119–171.

[30] L. Hörmander; The Analysis of linear partial differential operators, Vol III, Springer-Verlag, Berlin, Heidelberg, 1983.

[31] V. Isakov; *Completeness of products of solutions and some inverse problems for PDE*, Journal of Differential Equations, **92**, (1991), no. 2, 305–316.

[32] C. E. Kenig and M. Salo; *The Calderón problem with partial data on manifolds and applications*, Anal. PDE, **6** (2013), 2003–2048.

[33] Y. Kian; *Recovery of time-dependent damping coefficients and potentials appearing in wave equations from partial data*, SIAM J. Math. Anal., **48**(6):4021–4046, 2016.

[34] Y. Kian and L. Oksanen; *Recovery of time-dependent coefficient on Riemannian manifold for hyperbolic equations*, Int. Math. Res. Not. IMRN **2019**, no. 16, 5087–5126.

[35] Y. Kian; *Unique determination of a time-dependent potential for wave equations from partial data*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), no. 4, 973-990.

[36] Y. Kian and E. Soccorsi; *Hölder stably determining the time-dependent electromagnetic potential of the Schrödinger equation*, *SIAM J. Math. Anal.* **51**, (2019), no. 2, 627-647.

[37] Y. Kian and A. Tetlow; *Hölder Stable Recovery of Time-Dependent Electromagnetic Potentials Appearing in a Dynamical Anisotropic Schrödinger Equation*, Inverse Probl. Imaging, **14** (2020), no. 5, 819-839.

[38] K. Knudsen and M. Salo; *Determining nonsmooth first order terms from partial boundary measurements*; Inverse Probl. Imaging, **1** (2007), no. 2, 349–369.

[39] V.P. Krishnan and M. Vashisth; *An inverse problem for the relativistic Schrödinger equation with partial boundary data*, Applicable Analysis (2020), Vol. **99**, No. 11, 1889-1909.

[40] K. Krupchyk and G. Uhlmann, Uniqueness in an inverse boundary problem for a magnetic Schrodinger operator with a bounded magnetic potential, Comm. Math. Phys., 327 (2014), 993-1009.

[41] K. Krupchyk and G. Uhlmann; *Inverse problems for advection diffusion equations in admissible geometries*, Comm. Partial Differential Equations, Vol. **43** (2018), no. 4, 585–615.

[42] R. K. Mishra and M. Vashisth; *Determining the time dependent matrix potential in a wave equation from partial boundary data*, Applicable Analysis Volume **100**, (2021) No. 16, 3492-3508.

[43] J.-L. Lions and E. Magenes; *Problèmes aux limites non homogènes et applications*, Vol. 1 Travaux et Recherches Mathématiques, No. 17, Dunod, Paris, 1968, x3+ 372 pp.

[44] A.I. Nachman; *Global uniqueness for a two-dimensional inverse boundary value problem*, Ann. of Math. (2) **143** (1996), no. 1, 71–96.

[45] G. Nakamura and S. Sasayama; *Inverse boundary value problem for the heat equation with discontinuous coefficients*, J. Inverse Ill-Posed Probl., **21** (2013), no. 2, 217-232.

[46] G. Nakamura, Z.Q. Sun and G. Uhlmann; *Global identifiability for an inverse problem for the Schrödinger equation in a magnetic field* Math. Ann. **303** (1995), no. 3, 377–388.

[47] V. Pohjola; *A uniqueness result for an inverse problem of the steady state convection-diffusion equation*, SIAM J. Math. Anal., **47** (2015), 2084-2103.

[48] Rakesh and W. W. Symes; *Uniqueness for an inverse problem for the wave equation*, Comm. Partial Differential Equations, **13** (1988), 87–96.

[49] S.K. Sahoo and M. Vashisth; *A partial data inverse problem for the convection-diffusion equation*, Inverse Probl. Imaging **14** (2020), no. 1, 53–75.

[50] R. Salazar; *Determination of time-dependent coefficients for a hyperbolic inverse problem*, Inverse Problems, **29**(9):095015, 17, 2013.

[51] S. Senapati and M. Vashisth; *Stability estimate for a partial data inverse problem for the convection-diffusion equation*, Evolution Equations and Control Theory, Vol. **11**, No. 5, October 2022, pp. 1681–1699.

[52] Z. Sun; *An inverse boundary value problem for the Schrödinger operator with vector potentials*, Trans. Amer. Math. Soc., Vol. **338**. No. 2, (1993), 953-969.

[53] M. Spivak; *A comprehensive introduction to differential geometry*, Vol. I. Second edition, Publish or Perish, Inc., Wilmington, Del., 1979. xiv+668 pp.

[54] J. Sylvester and G. Uhlmann; *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math., **125** (1987), no. 1, 153–169.

[55] M. Zworski; Semiclassical Analysis, volume 138 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2012.

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